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SATELLITE ATTITUDE PREDICTION BY MULTIPLE TIME SCALES METHOD

by
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ABSTRACT

An investigation is made of the problem of predicting the attitude of satellites under the influence of external disturbing torques. The attitude dynamics are first expressed in a perturbation formulation which is then solved by the multiple scales approach. The independent variable, time, is extended into new scales, fast, slow, etc., and the integration is carried out separately in the new variables. The rapid and slow aspects of the dynamics are thus systematically separated, resulting in a more rapid computer implementation. The theory is applied to two different satellite configurations, rigid body and dual spin, each of which may have an asymmetric mass distribution. The disturbing torques considered are gravity gradient and geomagnetic. A comparison with conventional numerical integration shows that our approach is faster by an order of magnitude.

Finally, as multiple time scales approach separates slow and fast behaviors of satellite attitude motion, this property is used for the design of an attitude control device. A nutation damping control loop, using the geomagnetic torque for an earth pointing dual spin satellite, is designed in terms of the slow equation.

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CHAPTER 1

INTRODUCTION

1.1 General Background

The problem of predicting a satellite attitude motion, under the influence of its environmental torques, is of fundamental importance to many problems in space research. An example is the determination of required control torque as well as the amount of fuel or energy for the satellite attitude control devices. Similarly, a better prediction of the satellite attitude motion can be helpful in yielding more accurate data for many onboard experiments, such as the measurement of the geomagnetic field or the upper atmosphere density etc., which depend on the satellite orientation.

Yet, the problem of satellite attitude prediction is still one of the more difficult problems confronting space engineers today. Mathematically, the problem consists of integrating a set of non-linear differential equations with given initial conditions, such that the satellite attitude motion can be found as functions of time. However, the process of integrating these equations by a direct numerical method for long time intervals, such as hours, days (which could be even months or

years), is practically prohibited for reasons of computational cost and possible propagation of numerical round-off and truncation errors. On the other hand, it is even more difficult, if it is possible at all, to have an exact analytic solution of the problem, because of the non-linearity and the existence of various external disturbing torques in each circumstance.

A reasonable alternative approach to the above problem seems to be to apply an asymptotic technique for yielding an approximate solution. The purpose of this approach is to reduce the computational effort in the task of attitude prediction for long intervals, at the cost of introducing some asymptotic approximation errors. Meanwhile, the asymptotic approximate solution has to be numerically implemented in order to make it capable of handling a broad class of situations.

We found the problem is interesting and challenging in two ways. First,because it is basic, the results may have many applications. Second, the problem is very complicated, even an approximate approach is difficult both from analytic and numerical points of view.

1.2 Problem Description

The subject of this thesis is the prediction of satellite attitude motion under the influence of various disturbing torques. The main objective is to formulate a fast and accurate way of simulating the attitude rotational dynamics in terms of the angular velocity and Euler parameters as functions of time. The formulation has to be general, it must be able to handle any orbit, initial conditions, or satellite mass distribution. Furthermore, it must predict the long term secular effects and / or the complete attitude rotational motion, depending on the requirement. Because of this built-in generality it is intended that the program can be used as a design tool for many practical space engineering designs. To achieve this desired end the problem is first expressed as an Encke formulation. Then, the multiple time scales (MTS) technique is applied to obtain a uniformly valid asymptotic approximate solution to first order for the perturbed attitude dynamics.

Two different satellite configurations are considered, a rigid body satellite and a dual spin satellite, each of which may have an asymmetric mass distribution. In the latter case, it is assumed that the satellite contains a single fly wheel, mounted along one of the satellite

body-principal-axes, to stabilize the satellite attitude motion. These models are considered typical of many classes of satellites in operation today. The disturbing torques considered in this dissertation are the gravity gradient and the geomagnetic torques. For a high-orbit earth satellite these two torques are at least a hundred times bigger than any other possible disturbance, though, of course, there would be no difficulty inserting models of other perturbations.

Both the gravity gradient and the geomagnetic torques depend on the position as well as the attitude of the satellite with respect to the earth. Therefore, the orbital and attitude motion are slowly mixed by the actions of these disturbances. However, the attitude motion of the vehicle about its center of mass could occur at a much faster rate than the motion of the vehicle in orbit around the earth. Directly integrating this mixed motion, fast and slow together, is very inefficient in terms of computer time. However, realizing that there are these different rates, then the ratio of the averaged orbital angular velocity to the averaged attitude angular velocity or equivalently the ratio of the orbital and attitude frequencies (a small parameter denoted ε) may be used in the MTS technique to separate the slow orbital motion from the fast attitude motion. In this way the original

dynamics are replaced by two differential equations in terms of a slow and a fast time scale respectively. The secular effect as well as the orbit-attitude coupling is then given by the equation in the slow time scale, while the non-biased oscillatory motion is given by the second equation in terms of the fast time scale. In addition, a method for handling the resonance problem is also discussed.

In some situations the slow equation for the secular effects can be useful in the design of an attitude control system. The vehicle environment torques, if properly used, can be harnessed as a control force. However, to design such a control system it is often found that the control force is much too small to analyze the problem in the usual way. In fact, the design is facilitated in terms of the equation of the slow variable because only the long term secular motions can be affected. This application is demonstrated by mean of a nutation damping control loop using the geomagnetic torque.

1.3 Historical Review and Literature Survey

The problem of attitude dynamics of an artificial satellite, rigid body as well as dual spin case, is closely related to the branch of mechanics of rigid body rotational motion. The subject is regarded as one of the oldest branches of science, starting from middle of the eighteenth century. Since then it has interested many brilliant minds for generations. The literature in this area, therefore, is rich and vast. Thus, we have to focus our attention on only those areas which are immediately related to this research.

The classical approach to the rotational dynamics mainly seeks the analytic solutions and their geometric interpretations. By this approach, many important and elegant results have been obtained. Among them, the 'Poinsot construction' of L. Poinsot [11], gives a geometrical representation of the rigid body rotational motion. The Euler-Poinsot problem,for a torque-free motion, was first solved by G. Kirchhoff[12] by means of Jacobian elliptic functions. On the other hand, F. Klein and A. Sommerfeld [13], formulated the same problem in terms of the singularity-free Euler symmetric parameters and gave the solution. Recently, H. Morton, J. Junkins and others [15] solved the equations of Euler symmetric parameters again by introducing a set of complex orientation parameters. Kirchhoff's solution as well as

the solution of Euler symmetric parameters by H. Morton etc. play an important role- as a reference trajectory- in our study.

This approach, however, because of its nature can not handle a general problem for various situations. This difficulty is substantial for the case of an artificial satellite. A more flexible alternative, widely applied in the engineering world, is the asymptotic technique for evaluating an approximate solution. Among them, the averaging method by N. N. Bogoliubov and Y. Mitropolsky [14] is the most commonly used. For example Holland and Sperling[16] have used the averaging method for estimating the slow variational motion of the satellite angular momentum vector under the influence of gravity gradient torque and Beletskii [17] formulated perturbation equations using the osculating elements for a dynamically symmetric satellite. F. L. Chernous'ko [18] derived the equations of variation of parameters for angular momentum vector and the rotational kinetic energy for an asymmetric satellite. However, the averaging method is most easily applied for a problem which normally has a set of constant parameters, such that the slow variational behavior of these parameters can be established in a perturbed situation. For example in a simple harmonic oscillator, the frequency and amplitude are two parameters which characterize the dynamics described by a

second order differential equation. Unfortunately, rotational motion in general,does not immediately lead to a complete set of similar parameters. Although, it has constant angular momentum vector and kinetic energy as parameters it is a six-dimensional problem. Besides, an elliptic integral is involed in its solution. Nevertheless, this difficulty can be overcome by casting the problem in a Hamilton-Jacobi form,from which a variation-of-parameter formulation can be derived in terms of Jacobi elements. This approach is reflected in the works of Hitzl and Breakwell [19] Cochran[20], Pringle[21] etc.

Our dissertation is different from the others mainly in three aspects. First, it is a new approach, using the multiple time-scales method [1-7] with the Encke perturbation formulation [22], for predicting the complete satellite attitude motion without involving the Hamilton-Jacobi equation. Second, we are interested in the secular effect of the disturbing torques as well as the non-secular oscillatory effect. By combining them, we have the complete solution. Further we know that the former gives the long term behavior and the latter indicates the high-frequency motion of the satellite attitude dynamics. Third, our immediate objective is numerically oriented for saving computer time. Thus the difficulties we encounter could be analytical as well as numerical .

1.4 Arrangement of the Dissertation

Chapter 2 reviews the multiple time scales asymptotic technique - a basic tool in this research. Two examples are used for illustrating the fundamental procedure,one represents the secular type of almost-linear problem and the other represents the singular type of slowly time-varying linear system.

Chapter 3 develops the asymptotic solution to the attitude motion of a rigid body satellite under the influence of known small external torques. It shows that, the original equations of attitude dynamics can be represented by two separate equations - one describing the slow secular effects, and the other describing the fast oscillatory motion. The latter can be analytically evaluated. Numerical simulation using this approach is also presented for the class of rigid body satellites under the influence of gravity gradient and geomagnetic torques.

In chapter 4, the previous results are extended to the case of dual spin satellite, in which a fly-wheel is mounted onboard. Two sets of numerical simulations, one for a dual-spin satellite in the earth gravity gradient field and the other influenced by the geomagnetic field, are given.

Chapter 5 represents an application of the fact that MTS method separates the slow and fast behaviors of a satellite attitude motion. We demonstrate that the slow equation, which describes the secular effects, can be useful in the design of a satellite attitude control system. A nutation damping feedback control loop, using the geomagnetic torque for an earth pointing dual-spin satellite, is designed in terms of the slow equation.

In chapter 6 the conclusions drawn from the results of this study are summarized, and some suggestions for future research are listed.

CHAPTER 2

REVIEW OF MULTIPLE TIME SCALES (MTS) METHOD

2.1 Introduction

multiple time scales (MTS) method is one of the relatively newly developed asymptotic techniques. It enables us to develop approximate solutions to some complicated problems involving a small parameter ε, when the exact solutions are difficult, if not impossible, to find. The basic concept of MTS method is to extend the independent variable, usually time, into multi-dimensions. They are then used together with the expansion of the solution (dependent variable) such that an extra degree of freedom is created and the artificial secular terms can be removed. Thus a uniformly valid approximate solution is obtained [1-7].

An unique feature of the MTS method is that it can handle secular type as well as singular type of perturbation problems in a unified approach. By this method, the fast and slow behaviors of the dynamics are systematically identified and separated. The rapid motion is given in terms of a fast time scale and the slow motion in terms of a slow time scale, each of which, in most cases, has a meaningful physical explanation. A comprehensive reference on this subject is by Ramnath[3]. The textbook by Nayfeh [7] has also been found informative.

2.2 MTS and Secular Type of Problems

A secular perturbation problem is one in which the nonuniformity in a direct expansion occurs for large values of the independent variable. We consider systems with a small forcing term. The forcing term changes the dynamics gradually and has no appreciable effect in a short time. However, the long time secular effect of the small forcing term may significantly influence the overall behavior of the dynamics. From a mathematical point of view, a secular type of problem has a singularity at infinty in the time domain.

Since perturbation problems and the asymptotic technique for solving them can be most easily understood by solving a demonstration case, let us consider a simple example of a slowly damped linear oscillator [7],

$$\ddot{x} + x = -2\varepsilon\dot{x} \quad ; \quad 0 < \varepsilon << 1 \tag{2.2.1}$$

where ε is a small parameter. The simplicity of the forcing term $-2\varepsilon\dot{x}$ allows us to interpret the approximate solution, developed later. The exact solution is available, but the generality of our asymptotic approach will not be lost in spite of the simple form of the forcing term.

We first solve the problem by Poincaré type of direct expansion method [33], such that difficulties of non-uniformity and secular terms can be illustrated. Then the same problem is solved by MTS method, which

yields a uniformly valid asymptotic solution to first order.

We expand $x(t)$ into an asymptotic series in ε:

$$
\begin{aligned}
x(t) &= S_0 + S_1 + S_2 + \cdots \\
&= x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots\cdots \qquad (2.2.2)
\end{aligned}
$$

An asymptotic series is defined [9] as one in which the magnitude of each term is at least one order less than its previous one [i.e. $|S_n|/|S_{n-1}| = |\varepsilon^n x_n|/|\varepsilon^{n-1}x_{n-1}| \sim \varepsilon$]. Therefore S_i decreases rapidly as the index i increases. This simple fact allows us to approximate the solution by calculating only a few leading terms in the series expansion.

Substituting (2.1.2) into (2.1.1) and equating the coefficients of like powers of ε, we have:

ε^0 :

$$\ddot{x}_0 + x_0 = 0 \qquad (2.2.3)$$

ε^1 :

$$\ddot{x}_1 + x_1 = -2\dot{x}_0 \qquad (2.2.4)$$

$\vdots$

etc

The solution for $x_0(t)$ in equation (2.2.3) is

$$x_0 = a \cos t + b \sin t \tag{2.2.5}$$

Where 'a' and 'b' are two constants. Substituting x_0 into eq. (2.2.4) of x_1

$$\ddot{x}_1 + x_1 = -2\,(-a \sin t + b \cos t) \tag{2.2.6}$$

$$\text{with I.C.} \qquad x_1(0) = \dot{x}_1(0) = 0$$

The solution is :

$$x_1 = -a t \cos t - b t \sin t + a \sin t \tag{2.2.7}$$

The approximation of x(t) up to first order is therefore:

$$\begin{aligned} x_{app}(t) &= x_0(t) + \varepsilon x_1(t) \\ &= (a \cos t + b \sin t) \\ &\quad + \varepsilon\,(-a t \cos t - b t \sin t + a \sin t) \end{aligned} \tag{2.2.8}$$

Above approximation, however, is a poor one because of the occurrence of two terms $-a t \varepsilon \cos t$ and $-b t \varepsilon \sin t$, which approach infinity as $t \to \infty$. They are referred to as secular terms. We know the true x(t) has to be bounded and asymptotically decaying, for x(t) is a damped harmonic oscillator. The secular terms

make the series expansion $x(t) = x_0 + \varepsilon x_1 + \cdots$ in eq. (2.2.8) a non-asymptotic one, since $\frac{S_1}{S_0} = \frac{\varepsilon x_1}{x_0} \to \infty$ as $t \to \infty$. In the process of finding a solution by series expansion, there is no guarantee that the higher order terms can be ignored in a non-asymptotic series expansion. On the other hand, if an asymptotic series is truncated, the error due to the ignored higher order terms will be uniformly bounded in a sense that |error| / |approximate solution| $\sim \varepsilon$. An approximate solution is said to be 'uniformly valid' if its error is uniformly bounded in the interval of interest. We see that the loss of accuracy by straightforward Poincaré type of expansion is due to the occurrence of the secular terms and therefore the approximation is not uniformly valid.

In the following, the same problem will be studied in the context of multiple time scale method, which yields a uniformly valid solution.

For convenience, we rewrite the dynamics

$$\ddot{x} + x = -2\varepsilon \dot{x} \tag{2.2.9}$$

The solution $x(t)$ is first expanded into an asymptotic series of ε, same as before

$$x(t) = x_0(t) + \varepsilon x_1(t) + \cdots \tag{2.2.10}$$

The concept of extension is then invoked, which

expands the domain of the independent variable into a space of many independent variables. These new independent variables as well as the new terms that arise due to extension are then so chosen that the non-uniformities of direct perturbation can be eliminated [3].

Let

$$t \rightarrow [\tau_0, \tau_1, \tau_2 \cdots\cdots] \qquad (2.2.11)$$

For an almost linear problem, as the one we are studying

$$\tau_0 = t$$

$$\tau_1 = \varepsilon t$$

$$\tau_2 = \varepsilon^2 t$$

$$\vdots$$

etc

The new time scales, in general, can be non-linear as well as complex, which depend upon the nature of the problem [4]. However, for this particular problem, a set of simple linear time scales works just as well.

The time derivative can be extended in the space by partial derivatives as follows.

$$\frac{d}{dt} = \frac{\partial}{\partial \tau_0}\frac{d\tau_0}{dt} + \frac{\partial}{\partial \tau_1}\frac{d\tau_1}{dt} + \frac{\partial}{\partial \tau_2}\frac{d\tau_2}{dt} + \cdots$$

$$= \frac{\partial}{\partial \tau_0} + \varepsilon\frac{\partial}{\partial \tau_1} + \varepsilon^2\frac{\partial}{\partial \tau_2} + \cdots \qquad (2.2.12)$$

And

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial \tau_0^2} + 2\varepsilon \frac{\partial^2}{\partial \tau_0 \partial \tau_1} + \varepsilon^2 \left(\frac{\partial^2}{\partial \tau_0 \partial \tau_2} + \frac{\partial^2}{\partial \tau_1^2} \right) \tag{2.2.13}$$

Substituting equations (2.2.12), (2.2.13) and (2.2.10) into equation (2.2.9), and equating coefficients of like powers of ε , we have

ε^0 :

$$\frac{\partial^2 x_0}{\partial \tau_0^2} + x_0 = 0 \tag{2.2.14}$$

ε^1 :

$$\frac{\partial^2 x_1}{\partial \tau_0^2} + x_1 = -2 \frac{\partial x_0}{\partial \tau_0} - 2 \frac{\partial^2 x_0}{\partial \tau_0 \partial \tau_1}$$

$$\vdots \tag{2.2.15}$$

etc

The original equation has been replaced by a set of partial differential equations. The solution for $x_0(t)$ from (2.2.14) is

$$x_0 = a(\tau_1, \tau_2 \cdot\ ) \exp(i\tau_0) + b(\tau_1, \tau_2 \cdots) \exp(-i\tau_0) \tag{2.2.16}$$

where a,b are functions of τ_1, τ_2 ,...etc., and are yet to be determined. Substitute x_0 from (2.2.16) into (2.2.15), and solve for x_1 .

$$\frac{\partial^2 x_1}{\partial \tau_0^2} + x_1 = \exp(i\tau_0)\left(-2ia - 2i\frac{\partial a}{\partial \tau_1}\right) + \exp(-i\tau_0)\left(2bi + 2i\frac{\partial b}{\partial \tau_1}\right) \tag{2.2.17}$$

Equation (2.2.17) represents a harmonic oscillator driven by an external sinusoidal function. The $x_1(t)$ could become unlimited, since the external force has the same natural frequency as the system itself. In order to have $|x_1/x_0|$ bounded, the terms to the right hand side of the equal sign in (2.2.17) have to be set to zero, which will result in a bounded $x_1(t)$. Note that, this is possible, because there is a freedom of selecting functions 'a' and 'b' from the extension of the independent variable. By doing so, we have,

$$\begin{cases} 2a + 2\dfrac{\partial a}{\partial \tau_1} = 0 \\ 2b + 2\dfrac{\partial b}{\partial \tau_1} = 0 \end{cases} \tag{2.2.18}$$

or

$$\begin{cases} a = a_0 \exp(-\tau_1) \\ b = b_0 \exp(-\tau_1) \end{cases} \tag{2.2.19}$$

where a_0 and b_0 are two constants. Combining (2.2.16) and (2.2.19), the approximate solution for $x(t)$ up to first order of ε by MTS method is

$$\begin{aligned} x(t) &= x_0 + \varepsilon x_1 \\ &= a_0 \exp(it - \varepsilon t) + b_0 \exp(-it - \varepsilon t) \end{aligned} \tag{2.2.20}$$

The exact solution for (2.2.9) can be obtained, which is

$$x_{exact}(t) = a_0 \exp(i\sqrt{1-\varepsilon^2}\, t - \varepsilon t) + b_0 \exp(-i\sqrt{1-\varepsilon^2}\, t - \varepsilon t) \qquad (2.2.21)$$

The error/solution ratio in this case is

$$\left| \frac{\text{error}}{\text{Solution}} \right| = \left| \frac{x_{exact} - x_{approx}}{x_{exact}} \right| = O(\varepsilon^2 t)$$

It is interesting to note that by the MTS method, we have replaced the original dynamics (2.2.9) by an equation (2.2.14) in the fast time scale τ_0,

$$\frac{\partial^2 x_0}{\partial \tau_0^2} + x_0 = 0 \qquad (2.2.14)$$

and two slow equations in the slow time scale τ_1.

$$\frac{\partial a}{\partial \tau_1} + a = 0$$
$$\frac{\partial b}{\partial \tau_1} + b = 0 \qquad (2.2.18)$$

The fast equation gives the undisturbed oscillatory motion and the slow equations represent the slow variational change of the amplitude of the oscillation caused

by the damping term.

2.3 MTS Method and Singular Perturbation Problems

There is a class of perturbation problems in which the behavior of the reduced system - by setting ε equal to zero - could be dramatically different from the original. This phenomenon occurs because the reduced system, described by a lower order differential equation can not satisfy the given boundary conditions in general. We call this kind of problem a singular perturbation problem.

Singular perturbation problems have played an important role in the engineering field, most notably, in fluid mechanics, for eg. the boundary layer theory. This problem was solved by introducing the inner (Prandtl's boundary layer) and outer expansion [32]. However, the same problem also can be solved in a more straightforward approach by the MTS method. This approach was first noted in the paper by Ramnath [4] in studying the behavior of a slowly time-variant linear system by employing non-linear time scales. In the following, let us use an example, which is adopted from [3], for demonstration.

Consider a second order singular perturbation problem

$$\varepsilon^2 \ddot{y} + \omega(t)\, y = 0 \qquad (2.3.1)$$

Where $0 < \varepsilon << 1$, ε is a constant small parameter. Expand the time domain t into two-dimensions

$$t \rightarrow [\tau_0, \tau_1]$$

and define τ_0, τ_1 as follows

$$\tau_0 = t$$
$$\tau_1 = \frac{1}{\varepsilon}\int_{t_c}^{t} k(t)\, dt \qquad (2.3.2)$$

where $k(t)$ is yet to be determined.

The time derivative $\frac{d}{dt}$ and $\frac{d^2}{dt^2}$ can be extended as

$$\frac{d}{dt} = \frac{1}{\varepsilon}\frac{\partial}{\partial \tau_1} \cdot k(\tau_0) + \frac{\partial}{\partial \tau_0}$$

$$\frac{d^2}{dt^2} = \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial \tau_1^2} k^2(\tau_0) + 2\frac{1}{\varepsilon}\frac{\partial^2}{\partial \tau_0 \partial \tau_1} k(\tau_0) + \frac{1}{\varepsilon}\frac{\partial}{\partial \tau_1}\frac{\partial k}{\partial \tau_0} + \frac{\partial^2}{\partial \tau_0^2} \qquad (2.3.3)$$

Substituting (2.3.3) into (2.3.1) and separating terms according to the pover of ε , we will have the set of equations,

ε^0 :

$$k^2 \frac{\partial^2 y}{\partial \tau_1^2} + \omega(\tau_0)\, y = 0 \tag{2.3.4}$$

ε^1 :

$$\dot{k} \frac{\partial y}{\partial \tau_1} + 2k \frac{\partial^2 y}{\partial \tau_0 \partial \tau_1} = 0 \tag{2.3.5}$$

ε^2 :

$$\frac{\partial^2 y}{\partial \tau_0^2} = 0 \tag{2.3.6}$$

By assuming that $y(\tau_0, \tau_1)$ has a solution in the form

$$y(\tau_0, \tau_1) = \alpha(\tau_0) \exp(\tau_1) \tag{2.3.7}$$

substitution into (2.3.4), yields

$$k^2(\tau_0) + \omega(\tau_0) = 0$$

$$\text{or} \quad k_1 = i\,\omega^{\frac{1}{2}}$$
$$k_2 = -i\,\omega^{\frac{1}{2}} \tag{2.3.8}$$

Similarly put (2.3.7) into (2.3.5), we have

$$\frac{\partial k}{\partial \tau_0} \alpha + 2k \frac{\partial \alpha}{\partial \tau_0} = 0 \tag{2.3.9}$$

or

$$\alpha = k^{-\frac{1}{2}} \tag{2.3.10}$$

The approximate solution up to first order can be constructed as

$$y(t) = C_1' k_1^{-\frac{1}{2}} \exp\left[\frac{1}{\varepsilon}\int k_1 dt\right] + C_2' k_2^{-\frac{1}{2}} \exp\left[\frac{1}{\varepsilon}\int k_2 dt\right]$$

$$= C_1 \omega^{-\frac{1}{4}} \exp\left[\frac{i}{\varepsilon}\int \omega^{\frac{1}{2}} dt\right] + C_2 \omega^{-\frac{1}{4}} \exp\left[\frac{-i}{\varepsilon}\int \omega^{\frac{1}{2}} dt\right]$$

(2.3.11)

We obtain (2.3.11) from (2.3.8) and (2.3.10). Note that in our approximation the frequency variation is described on the τ_1 scale and the amplitude variation on the τ_0 scale. The success of this approach depends on the proper choice of the nonlinear clock. While in the past this choice was made on intuitive grounds, recent work [2] has been directed towards a systematic determation of the clocks. Ramnath [3] has shown that the best nonlinear clocks can be determined purely in a deductive manner, from the governing equations of the system. With a judicious choice of scales the accuracy of the asymptotic approximation is assured. A detailed error analysis of the approximation was given by Ramnath [3]. These questions are beyond the scope of the present effort and reference [3] may be consulted for more information.

CHAPTER 3

PREDICTION OF ATTITUDE MOTION FOR A RIGID BODY SATELLITE

3.1 Introduction

In this chapter, a multiple time scales asymptotic technique is applied for the prediction of a rigid body satellite attitude motion disturbed by a small external torque.

The attitude dynamics of a satellite, described in terms of the Euler's equations and Euler symmetric parameters, are first perturbed into an Encke formulation, in which the torque-free case is considered as a nominal solution. The multiple time scales technique is then used for the separation of the fast attitude motion from the slow orbital motion in an approximate, but asymptotic way. Thereby, the original dynamics can be replaced by two sets of partial differential equations in terms of a slow and a fast time scale. The long-term secular effects due to the disturbing torque are given by the equations in the slow time scale, which operate at the same rate as the orbital motion. A non-biased oscillatory motion is given by the second set of equations in terms of the fast time scale, which basically describes the vehicle attitude oscillatory motion. These fast and slow motions, combined,

give us a first order asymptotic solution to the Encke's perturbational equation, which, therefore, can be regarded as the second order asymptotic solution to the original satellite attitude dynamics.

Finally, the fast non-biased oscillatory motion can be analytically evaluated if the external torques are not explicitly functions of time. Thus numerical simulation of a satellite rotational motion by this new approach requires only the integration of the slow equation which can be done with a large integration time step. This fact leads to a significant saving of computer time as compared to a direct numerical integration. Two examples, one with gravity gradient torque the other with geomagnetic torque, are demonstrated in section 3.6 and 3.7.

3.2 Rigid Body Rotational Dynamics

(A). Euler's Equations

Newton's second law for rigid body rotational motion in an inertial frame can be written as

$$\frac{d\bar{H}^{\wedge}}{dt} = \bar{M}^{\wedge} \tag{3.2.1}$$

Where $\bar{H}$ and $\bar{M}$ are the angular momentum and the external torque. By Coriolis law, the motion can be

expressed in any moving frame 'b', as:

$$\frac{d\bar{H}^b}{dt} + \bar{\omega}_{Ib}^b \times \bar{H}^b = \bar{M}^b \tag{3.2.2}$$

Where $\bar{\omega}_{ib}$ is the angular velocity of the 'b' frame with respect to the inertial frame. In case the 'b' frame is selected to coincide with the body fixed principal axes (x,y,z), then the angular momentum can be written as

$$\bar{H}^b = \overset{*}{I}_m \overset{*}{\omega}{}_{Ib}^b = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} I_x \omega_x \\ I_y \omega_y \\ I_z \omega_z \end{bmatrix} \tag{3.2.3}$$

where I_x, I_y, I_z are moments of inertia about x, y, z axes. Combining (3.2.2) and (3.2.3) we have Euler's equations

$$\begin{aligned} I_x \dot{\omega}_x - (I_y - I_z)\,\omega_y \omega_z &= M_x \\ I_y \dot{\omega}_y - (I_z - I_x)\,\omega_z \omega_x &= M_y \\ I_z \dot{\omega}_z - (I_x - I_y)\,\omega_x \omega_y &= M_z \end{aligned} \tag{3.2.4}$$

In vector notation, they are :

$$\overset{*}{I}_m \dot{\bar{\omega}} + (\bar{\omega} \times) \overset{*}{I}_m \bar{\omega} = \bar{M} \qquad (3.2.5)$$

Euler's equations give the angular velocity of a rigid body with respect to an inertial space though this angular velocity is expressed in the instantaneous body fixed principal axes [35].

(B). Euler Symmetric Parameters

The role of Euler symmetric parameters are similar to Euler angles, which define the relative orientation between two coordinates. From either of them, a transformation matrix can be calculated and a vector can be transformed from one coordinate to another by pre-multiplying with the transformation matrix.

However, from an application point of view, there are notable differences between Euler symmetric parameters and Euler angles. The important ones are listed as follows.

1. Euler angles (θ_i , $i = 1, 2, 3$) have order of three, whereas Euler symmetric parameters (β_i , $i=0,1,2,3$) have order of four with one constraint.
2. β_i are free from singularity, where θ_i are not. Since θ_1 , θ_2 , θ_3 are the z-x-z rotations, in case that $\theta_2 = 0$, one can not distinguish θ_1

from θ_3 .

3. β_i propagate by a linear homogenous differential equation; θ_i are by a non-linear differential equation.

4. θ_i have a clear physical interpretation, -i.e. precession, nutation and rotation. β_i can not be immediately visualized.

By considering above differences, we feel that Euler symmetric parameters are more suitable for numerical computation because they are propagated by a linear equation and free from singularity, even though they add one more dimension to the problem. On the other hand, Euler angles are easier to understand. In this chapter we select Euler symmetric parameters for the satellite attitude prediction problem. And in chapter 5, for a satellite attitude control system design, we use Euler angles.

The concept of Euler symmetric parameters is based upon 'Euler Theorem', which says that a completely general angular displacement of a rigid body can be accomplished by a single rotation ψ about a unit vector $\hat{\ell} = (\ell_1, \ell_2, \ell_3)$, where $\hat{\ell}$ is fixed to both body and reference frames. The β_i are then defined as :

$$\beta_0 \triangleq \cos\frac{\psi}{2}$$
$$\beta_i \triangleq l_i \sin\frac{\psi}{2} \qquad ; \qquad i = 1, 2, 3 \qquad (3.2.6)$$

with a constraint

$$\sum_{i=0}^{3} \beta_i^2 = 1 \qquad (3.2.7)$$

If $\overset{*}{C}_{rb}$ is the transformation matrix from the reference frame 'r' to the frame 'b',

$$\bar{V}_b = \overset{*}{C}_{rb} \; \bar{V}_r$$

Then $\overset{*}{C}_{rb}$ can be calculated in term of β_i as [35]

$$\overset{*}{C}_{rb} = \begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2, & 2(\beta_1\beta_2+\beta_0\beta_3), & 2(\beta_1\beta_3-\beta_0\beta_2) \\ 2(\beta_1\beta_2-\beta_0\beta_3), & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2, & 2(\beta_2\beta_3+\beta_0\beta_1) \\ 2(\beta_1\beta_3+\beta_0\beta_2), & 2(\beta_2\beta_3-\beta_0\beta_1), & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix}$$

(3.2.8)

Also β_i satisfy a homogeneous linear differential equation [35] ;

$$\frac{d}{dt}\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix}\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \qquad (3.2.9)$$

where ω_x is the component of $\bar{\omega}_{rb}^{b}$ in x direction, etc.

3.3 Euler-Poinsot Problem

Closed form solutions for a rigid body rotational motion with external torque are usually not possible except for a few special cases. A particular one, named after Euler and Poinsot, is the zero external torque case. This is useful here, since the disturbing torques acting on a satellite are small, the Euler-Poinsot case can be taken as a nominal trajectory.

It is Kirchhoff [12] who first derived the complete analytic solution for Euler's equation ($\bar{\omega}$) in terms of time, in which an elliptic integral of the first kind is involved. In the following, we will review the Kirchhoff's solution along with the solution of Euler symmetric parameters ($\bar{\beta}$) by Morton and Junkins [15]. Also, by defining a polhode frequency, we find that the solution for β_i can be further simplified, such that it contains only periodic functions.

(A) Kirchhoff's Solution

Without external torque, Euler's equations are

$$I_x \dot{\omega}_x - (I_y - I_z)\,\omega_y \omega_z = 0 \tag{3.3.1}$$

$$I_y \dot{\omega}_y - (I_z - I_x)\,\omega_z \omega_x = 0 \tag{3.3.2}$$

$$I_z \dot{\omega}_z - (I_x - I_y)\,\omega_x \omega_y = 0 \tag{3.3.3}$$

Above equations are equivalent to a third order homogeneous ordinary differential equation and its solution involves three integration constants.

Multiplying above three equations by ω_x, ω_y and ω_z respectively, and integrating the sum, we obtain one of the integration constants for the problem, called T, which is the rotational kinetic energy of the system, that is:

$$I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 = 2T \tag{3.3.4}$$

Similarly, by multiplying the three Euler's equations by $I_x\omega_x$, $I_y\omega_y$ and $I_z\omega_z$ respectively and integrating the sum, we have H, another integration constant, that is the angular momentum of the system.

$$I_x^2 \omega_x^2 + I_y^2 \omega_y^2 + I_z^2 \omega_z^2 = H^2 \tag{3.3.5}$$

Having rotational energy T and angular momentum H given, a new variable ϕ can be defined in terms of time t by an elliptic integral of the first kind,

$$\lambda(t - \tau) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2(\phi)}} \tag{3.3.6}$$

where λ, τ and k are constants, and k is the modulus of the elliptic integral.

Kirchhoff's solution can be written as follows ;

$$\omega_x = a \ (1 - k^2 \sin^2 \phi)^{\frac{1}{2}} \tag{3.3.7}$$

$$\omega_y = b \ \sin \phi \tag{3.3.8}$$

$$\omega_z = c \ \cos \phi \tag{3.3.9}$$

where constants a,b,c,k,λ and τ are defined as follows.

$$a^2 = \frac{H^2 - 2 I_z T}{I_x (I_x - I_z)} \tag{3.3.10a}$$

$$b^2 = \frac{2 I_x T - H^2}{I_y (I_x - I_y)} \tag{3.3.10b}$$

$$c^2 = \frac{2 I_x T - H^2}{I_z (I_x - I_z)} \tag{3.3.10c}$$

$$\lambda^2 = \frac{(I_x - I_y)(H^2 - 2 I_z - T)}{I_x I_y I_z} \tag{3.3.10d}$$

$$k^2 = \frac{I_y - I_z}{I_x - I_y} \cdot \frac{2 I_x T - H^2}{H^2 - 2 I_z T} \tag{3.3.10e}$$

$$\tau = \frac{-1}{\lambda} \int_0^{\varphi_0} \frac{d\phi}{(1 - k^2 \sin^2 \phi)^{\frac{1}{2}}} \tag{3.3.10f}$$

$$\varphi_0 = \sin^{-1}\left[\frac{\omega_y(t_0)}{b}\right] \qquad (3.3.10g)$$

Signs of a,b,c and λ should be picked, such that they satisfy the equation

$$\frac{I_x - I_y}{I_z} = \frac{-c\lambda}{ab} \qquad (3.3.11)$$

The validity of above solution can be proved by direct substitution.

Kirchhoff's solution is less popular than Poinsot construction in the engineering world. One reason is that it involves an elliptic integral, the solution becomes rather unfamiliar to the engineering analyst. However, for our long term satellite attitude prediction problem, Kirchhoff's solution seems to be a powerful tool.

(B) <u>Solution For Euler Symmetric Parameters</u>

Euler symmetric parameters satisfy a linear differential equation, which relates ω_i and β_i [35]

$$\frac{d}{dt}\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \beta_3\end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix}\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \beta_3\end{bmatrix} \qquad (3.3.12)$$

The constraint is

$$\sum \beta_i^2 = 1$$

A set of complex numbers $\alpha_i(t)$ can be introduced as follows

$$\begin{aligned}
\alpha_0 &= \beta_0 + i\,\beta_2 \\
\alpha_1 &= -\beta_3 + i\,\beta_1 \\
\alpha_2 &= \beta_0 - i\,\beta_2 \\
\alpha_3 &= \beta_3 + i\,\beta_1
\end{aligned} \qquad (3.3.13)$$

where

$$i = \sqrt{-1}$$

and α_i satisfy a constraint of

$$\alpha_0\,\alpha_2 - \alpha_1\,\alpha_3 = 1 \qquad (3.3.14)$$

In matrix and vector notations, (3.3.12) and (3.3.13) are

$$\dot{\bar{\beta}} = \frac{1}{2}\,[\overset{*}{\omega}]\,\bar{\beta} \qquad (3.3.15)$$

$$\bar{\alpha} = \overset{*}{A}\,\bar{\beta} \qquad (3.3.16)$$

where

$$
[\overset{*}{\omega}] = \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix}
$$

and

$$
\overset{*}{A} = \begin{bmatrix} 1 & 0 & i & 0 \\ 0 & i & 0 & -1 \\ 1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \end{bmatrix}
$$

Substituting (3.3.16) into (3.3.15), we obtain:

$$
\dot{\bar{\alpha}} = \frac{1}{2}\ \overset{*}{A}\ [\overset{*}{\omega}]\ \overset{*}{A}{}^{-1}\bar{\alpha} \tag{3.3.17}
$$

or

$$
\begin{aligned}
\dot{\alpha}_0 &= \frac{1}{2}\left[\ i\ \omega_y + (i\ \omega_x - \omega_z)\frac{\alpha_3}{\alpha_0}\right]\alpha_0 \\
\dot{\alpha}_1 &= \frac{1}{2}\left[\ i\ \omega_y + (i\ \omega_x - \omega_z)\frac{\alpha_2}{\alpha_1}\right]\alpha_1 \\
\dot{\alpha}_2 &= \frac{1}{2}\left[-i\ \omega_y + (i\ \omega_x + \omega_z)\frac{\alpha_1}{\alpha_2}\right]\alpha_2 \\
\dot{\alpha}_3 &= \frac{1}{2}\left[-i\ \omega_y + (i\ \omega_x + \omega_z)\frac{\alpha_0}{\alpha_3}\right]\alpha_3
\end{aligned} \tag{3.3.18}
$$

For a torque - free rigid body rotation, the angular momentum vector $\bar{H}$ remains constant. If we assume one of our inertial axes (say $\hat{\lambda}_2$) pointing in $\bar{H}$ direction and name this particular inertial frame by 'n', then

$$\bar{H}^n = \begin{bmatrix} 0 \\ H \\ 0 \end{bmatrix} \tag{3.3.19}$$

Since

$$\bar{H}^b = \overset{*}{C}_{nb} \; \bar{H}^n \tag{3.3.20}$$

combining (3.3.19), (3.3. 20) and (3.2.8) we have:

$$\bar{H}^b = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = H \begin{bmatrix} 2\,(\beta_1\beta_2 + \beta_0\beta_3) \\ \beta_0^2 - \beta_1^2 + \beta_2^2 - \beta_3^2 \\ 2\,(\beta_2\beta_3 - \beta_0\beta_1) \end{bmatrix} \tag{3.3.21}$$

Using the relations

$$\bar{H}^b = \begin{bmatrix} I_x \;\, \omega_x \\ I_y \;\, \omega_y \\ I_z \;\, \omega_z \end{bmatrix} \qquad \text{and} \qquad \bar{\beta} = \overset{*}{A}{}^{-1} \bar{\alpha}$$

(3.3.21) can be written in ω_i and α_i as

$$
\begin{aligned}
I_x\, \omega_x &= H\, (-\alpha_0 \alpha_1 + \alpha_2 \alpha_3) \\
I_y\, \omega_y &= H\, (\alpha_0 \alpha_2 + \alpha_1 \alpha_3) \\
I_z\, \omega_z &= i H\, (\alpha_0 \alpha_1 + \alpha_2 \alpha_3)
\end{aligned}
\tag{3.3.22}
$$

Also α_i satisfy the constraint

$$
\alpha_0 \alpha_2 - \alpha_1 \alpha_3 = 1 \tag{3.3.23}
$$

As equations (3.3.22) and (3.3.23) are linear in $\alpha_0 \alpha_1$, $\alpha_0 \alpha_2$, $\alpha_1 \alpha_3$ and $\alpha_2 \alpha_3$, they can be solved in terms of $\omega_x, \omega_y, \omega_z$ and H, i.e.

$$
\begin{aligned}
\alpha_0\, \alpha_1 &= \frac{-1}{2H}\, (I_x\, \omega_x + i\, I_z\, \omega_z) \\
\alpha_0\, \alpha_2 &= \frac{1}{2}\, (I_y\, \frac{\omega_y}{H} + 1) \\
\alpha_1\, \alpha_3 &= \frac{1}{2}\, (\frac{I_y\, \omega_y}{H} - 1) \\
\alpha_2\, \alpha_3 &= \frac{1}{2H}\, (-i\, I_z + I_x\, \omega_x)
\end{aligned}
\tag{3.3.24}
$$

The ratio $\frac{\alpha_3}{\alpha_0}$ and $\frac{\alpha_2}{\alpha_1}$ can be easily calculated :

$$\frac{\alpha_3}{\alpha_0} = \frac{H - I_y \omega_y}{I_x \omega_x + i I_z \omega_z} = \frac{I_x \omega_x - i I_z \omega_z}{H + I_y \omega_y} \qquad (3.3.25)$$

$$\frac{\alpha_2}{\alpha_1} = \frac{-H - I_y \omega_y}{I_x \omega_x + i I_z \omega_z} = \frac{-I_x \omega_x + i I_z \omega_z}{H - I_y \omega_y} \qquad (3.3.26)$$

Substituting (3.3.25) and (3.3.26) into equation (3.3.18), we have

$$\frac{d\alpha_0}{dt} = \frac{1}{2}\left[i \, \frac{2T + H\omega_y}{H + I_y \omega_y} + \frac{(I_z - I_x)\,\omega_x \omega_z}{H + I_y \omega_y} \right] \alpha_0$$

$$\frac{d\alpha_1}{dt} = \frac{1}{2}\left[i \, \frac{-2T + H\omega_y}{H - I_y \omega_y} - \frac{(I_z - I_x)\,\omega_x \omega_z}{H - I_y \omega_y} \right] \alpha_1$$

$$\frac{d\alpha_2}{dt} = \frac{1}{2}\left[-i \, \frac{2T + H\omega_y}{H + I_y \omega_y} + \frac{(I_z - I_x)\,\omega_x \omega_z}{H + I_y \omega_y} \right] \alpha_2$$

$$\frac{d\alpha_3}{dt} = \frac{1}{2}\left[-i \, \frac{-2T + H\omega_y}{H - I_y \omega_y} - \frac{(I_z - I_x)\,\omega_x \omega_z}{H - I_y \omega_y} \right] \alpha_3$$

(3.3.27)

Now we have four decoupled, time-variant homogenous linear equations. Their solutions are immediately

available in terms of quadratures.

Also because (3.3.27) have periodic coefficients ($\omega_x, \omega_y, \omega_z$ are periodic), by Floquet theory [36], the solution can be expressed in the form $Q(t)\exp(st)$, where $Q(t)$ is a periodic function and s is a constant. With this in mind, solutions for (3.3.27) are

$$
\begin{aligned}
\alpha_0(t) &= E_1 \exp(i p_1) \exp(iRt)\, \alpha_0(t_0) \\
\alpha_1(t) &= E_2 \exp(-i p_2) \exp(-iRt)\, \alpha_1(t_0) \\
\alpha_2(t) &= E_1 \exp(-i p_1) \exp(-iRt)\, \alpha_2(t_0) \\
\alpha_3(t) &= E_2 \exp(i p_2) \exp(iRt)\, \alpha_3(t_0)
\end{aligned}
\tag{3.3.28}
$$

where

$$E_1 = \left[\frac{H + I_y \omega_y(t)}{H + I_y \omega_y(t_0)} \right]^{\frac{1}{2}}$$

$$E_2 = \left[\frac{H - I_y \omega_y(t)}{H - I_y \omega_y(t_0)} \right]^{\frac{1}{2}}$$

$$P_1 = \frac{1}{2} \left[\frac{2T}{H} - \frac{H}{I_y} \right] \cdot \left[\pi(t) - \frac{t}{T_\omega} \pi(T_\omega) - \frac{I_y}{H} \Psi(t) \right]$$

$$P_2 = \frac{1}{2}\left(\frac{2T}{H} - \frac{H}{I_y}\right)\left[\pi(t) - \frac{t}{T_\omega}\pi(T_\omega) + \frac{I_y}{H}\Psi(t)\right]$$

$$R = \frac{1}{2}\left[\frac{H}{I_y} + \left(\frac{2T}{H} - \frac{H}{I_y}\right)\frac{\pi(T_\omega)}{T_\omega}\right]$$

and

$$\pi(t) = \int_0^t \frac{dt}{1 - \frac{I_y^2}{H^2} b^2 \sin^2\phi}$$

$\Pi(t)$ is an elliptic integral of the third kind. T_ω is the period of ω_x, ω_y, and ω_z ; ϕ and t are related by equation (3.3.6). Also $\Psi(t)$ is given by

$$\Psi(t) = \int_0^t \frac{b \sin\phi \, dt}{1 - \frac{I_y^2 b^2}{H^2}\sin^2\phi}$$

Note there are two frequencies involved in the solution of Euler symmetric parameters : the first one is the same as the angular velocity ω_x , ω_y and ω_z with period of T_ω , and the second is related to $\exp(\pm iRt)$ with period of $\frac{2\pi}{R}$. The latter one can be explained as due to the motion of the axis of the instantaneous angular velocity vector $\tilde{\omega}_{1b}$. In Poinsot

construction, the tip of the $\bar{\omega}_{Ib}$ vector makes a locus on the invariant plane called the herpolhode and also a locus on the momentum ellipsoid called polhode. The time required for the vector $\bar{\omega}_{Ib}$ to complete a closed polhode locus is $\frac{2\pi}{R}$. We call 'R' the polhode frequency.

From $\bar{\beta} = \overset{*}{A}{}^{-1} \bar{\alpha}$, the general solution for $\bar{\beta}_i$ is:

$$\begin{bmatrix} \beta_0(t) \\ \beta_1(t) \\ \beta_2(t) \\ \beta_3(t) \end{bmatrix} = \begin{bmatrix} E_1 \cos(P_1+Rt), & 0, & -E_1 \sin(P_1+Rt), & 0 \\ 0, & E_2 \cos(P_2+Rt), & 0, & E_2 \sin(P_2+Rt) \\ E_1 \sin(P_1+Rt), & 0, & E_1 \cos(P_1+Rt), & 0 \\ 0, & -E_2 \sin(P_2+Rt), & 0, & E_2 \cos(P_2+Rt) \end{bmatrix} \begin{bmatrix} \beta_0(t_0) \\ \beta_1(t_0) \\ \beta_2(t_0) \\ \beta_3(t_0) \end{bmatrix}$$

(3.3.29)

(C) <u>The Transformation Matrix $\overset{*}{C}_{nb}$</u>

The transformation matrix $\overset{*}{C}_{nb}$ in terms of β_i is

$$\overset{*}{C}_{nb} = \begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2, & 2(\beta_1\beta_2+\beta_0\beta_3), & 2(\beta_1\beta_3-\beta_0\beta_2) \\ 2(\beta_1\beta_2-\beta_0\beta_3), & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2, & 2(\beta_3\beta_2+\beta_0\beta_1) \\ 2(\beta_1\beta_3+\beta_0\beta_2), & 2(\beta_2\beta_3-\beta_0\beta_1), & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix}$$

where βi is given by (3.3.29). By direct substitution, we have :

$$\overset{*}{C}_{nb} = \cos(2Rt)\,\overset{*}{C}_1{}' + \sin(2Rt)\,\overset{*}{C}_2{}' + \overset{*}{C}_3{}' \qquad (3.3.30)$$

where

$$\overset{*}{C}_1{}' = \begin{bmatrix} \begin{array}{l} \cos 2p_1\, E_1^2\, (\beta_0^2(0) - \beta_2^2(0)] \\ -2\sin 2p_1\, E_1^2\, \beta_0(0)\beta_2(0) \\ +\cos 2p_2\, E_2^2 [\beta_1^2(0) - \beta_3^2(0)] \\ +2\sin 2p_2\, E_2^2\, \beta_1(0)\beta_3(0) \end{array} & , \; 0, & \begin{array}{l} 2E_2^2 \beta_1(0)\beta_3(0)\cos 2p_2 \\ +\sin 2p_2\, E_2^2 [\beta_3^2(0) - \beta_1^2(0)] \\ -2\cos 2p_1\, E_1^2\, \beta_0(0)\beta_2(0) \\ -\sin 2p_1\, E_1^2 [\beta_0^2(0) - \beta_2^2(0)] \end{array} \\[2ex] \begin{array}{l} 2E_1E_2 \{\cos(p_1+p_2)[\beta_1(0) \\ \beta_2(0) - \beta_0(0)\beta_3(0)] \\ +\sin(p_1+p_2)[\beta_2(0)\beta_3(0) \\ +\beta_1(0)\beta_0(0)]\} \end{array} & , \; 0, & \begin{array}{l} 2E_1E_2\{\cos(p_1+p_2)[\beta_2(0)\beta_3(0) \\ +\beta_0(0)\beta_1(0)] + \sin(p_1+p_2) \\ [\beta_0(0)\beta_3(0) - \beta_1(0)\beta_2(0)]\} \end{array} \\[2ex] \begin{array}{l} E_2^2\{2\cos 2p_2\, \beta_1(0)\beta_3(0) \\ \quad +\sin 2p_2\, [\beta_3^2(0) - \beta_1^2(0)]\} \\ +E_1^2\{2\beta_0(0)\beta_2(0)\cos 2p_1 \\ \quad +\sin 2p_1 [\beta_0^2(0) - \beta_2^2(0)]\} \end{array} & , \; 0, & \begin{array}{l} \cos 2p_1\, E_1^2 [\beta_0^2(0) - \beta_2^2(0)] \\ -2\sin 2p_1\, E_1^2\, \beta_0(0)\beta_2(0) \\ +\cos 2p_2\, E_2^2 [\beta_3^2(0) - \beta_1^2(0)] \\ -2\sin 2p_2\, E_2^2\, \beta_1(0)\beta_3(0) \end{array} \end{bmatrix}$$

$$\overset{*}{C}_2{}' = \begin{bmatrix} \begin{array}{l} -\sin 2p_1\, E_1^2 \left(\beta_0^2(0)-\beta_2^2(0)\right) \\ -2\cos 2p_1\, E_1^2\, \beta_0(0)\,\beta_2(0) \\ -\sin 2p_2\, E_2^2 \left[\beta_1^2(0)-\beta_3^2(0)\right] \\ +2\cos 2p_2\, E_2^2\, \beta_1(0)\beta_3(0) \end{array} & , \; 0 \;, & \begin{array}{l} E_2^2\Big\{-2\sin 2p_2\, \beta_1(0)\beta_3(0) \\ \quad +\cos 2p_2 \left[\beta_3^2(0)-\beta_1^2(0)\right]\Big\} \\ +E_1^2\Big\{2\sin 2p_1\, \beta_0(0)\beta_2(0) \\ \quad -\cos 2p_1 \left[\beta_0^2(0)-\beta_2^2(0)\right]\Big\} \end{array} \\ & & \\ \begin{array}{l} 2E_1E_2\Big\{-\sin(p_1+p_2)\left[\beta_1(0)\beta_2(0)\right. \\ \left.-\beta_0(0)\beta_3(0)\right]+\cos(p_1+p_2) \\ \left[\beta_2(0)\beta_3(0)+\beta_1(0)\beta_0(0)\right]\Big\} \end{array} & , \; 0 \;, & \begin{array}{l} 2E_1E_2\Big\{-\sin(p_1+p_2)\left(\beta_2(0)\beta_3(0)\right. \\ \left.+\beta_0(0)\beta_1(0)\right]+\cos(p_1+p_2) \\ \left(\beta_0(0)\beta_3(0)-\beta_1(0)\beta_2(0)\right]\Big\} \end{array} \\ & & \\ \begin{array}{l} E_2^2\Big[-2\sin 2p_2\, \beta_1(0)\beta_3(0) \\ \quad +\cos 2p_2 \left[\beta_3^2(0)-\beta_1^2(0)\right]\Big] \\ +E_1^2\Big[-2\beta_0(0)\beta_2(0)\sin 2p_1 \\ \quad +\cos 2p_1 \left(\beta_0^2(0)-\beta_2^2(0)\right)\Big] \end{array} & , \; 0 \;, & \begin{array}{l} -\sin 2p_1\, E_1^2 \left[\beta_0^2(0)-\beta_2^2(0)\right] \\ -2\cos 2p_1\, E_1^2\, \beta_0(0)\beta_2(0) \\ -\sin 2p_2\, E_2^2 \left[\beta_3^2(0)-\beta_1^2(0)\right] \\ -2\cos 2p_2\, E_2^2\, \beta_1(0)\beta_3(0) \end{array} \end{bmatrix}$$

$$\overset{*}{C}_3{}' = \begin{bmatrix} 0 \,, & \begin{array}{l} 2E_1E_2\Big\{\cos(p_1-p_2)\left[\beta_1(0)\beta_2(0)+\beta_0(0)\beta_3(0)\right] \\ \quad +\sin(p_1-p_2)\left[\beta_1(0)\beta_0(0)-\beta_2(0)\beta_3(0)\right]\Big\} \end{array} & , \; 0 \\ & & \\ 0 \,, & \begin{array}{l} E_1^2\left[\beta_0^2(0)+\beta_2^2(0)\right] \\ \qquad -E_2^2\left[\beta_1^2(0)+\beta_3^2(0)\right] \end{array} & , \; 0 \\ & & \\ 0 \,, & \begin{array}{l} 2E_1E_2\Big\{\cos(p_1-p_2)\left[\beta_2(0)\beta_3(0)-\beta_0(0)\beta_1(0)\right] \\ \quad +\sin(p_1-p_2)\left[\beta_0(0)\beta_3(0)+\beta_1(0)\beta_2(0)\right]\Big\} \end{array} & , \; 0 \end{bmatrix}$$

Note that matrices $\overset{*}{C}'_1$, $\overset{*}{C}'_2$ and $\overset{*}{C}'_3$ are periodic functions with period of T_ω only.

Summary of the Section

1. Without external torque, both Euler's equations and Euler symmetric parameters can be solved analytically. The solutions are given by equations (3.3.7) to (3.3.10) and (3.3.29).
2. The angular velocity $\bar{\omega}(t)$ is a periodic function, that is $\bar{\omega}(t+T_\omega)=\bar{\omega}(t)$, whereas $\bar{\beta}(t)$ contains two different frequencies: they are the $\bar{\omega}$-frequency and the polhode-frequency.
3. The transformation matrix $\overset{*}{C}_{nb}$ is given by equation (3.3.30), in which the $\bar{\omega}$-frequency and the polhode-frequency are factored.

3.4 Disturbing Torques on a Satellite

Of all the possible disturbing torques which act on a satellite, the gravity gradient torque (G.G.T.) and the geomagnetic torque (G.M.T.) are, by far, the most important. Fig. 3.4.1 illustrates the order of magnitude of various passive external torques on a typical satellite [17], in which torques are plotted in terms of altitude. Note that, except for very low orbits, the G.G.T. and the G.M.T. are at least a hundred times as big as the others.

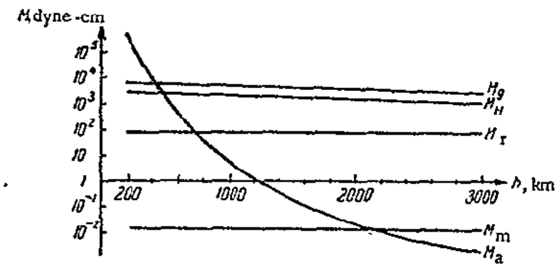


Figure 3.4.1.

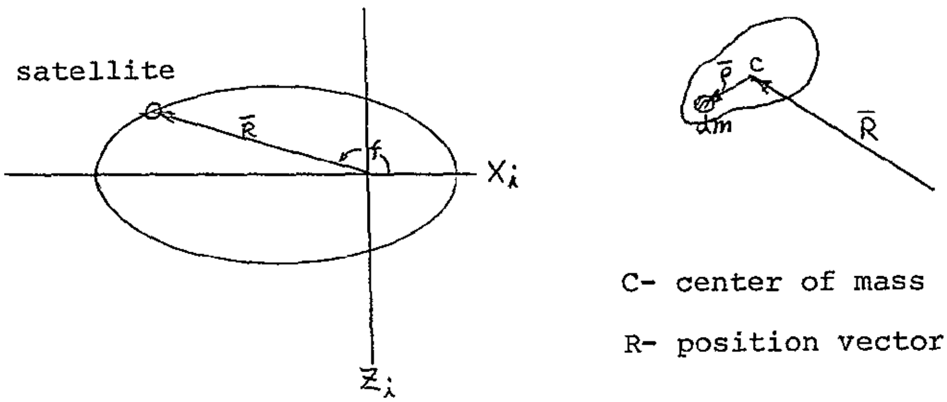
Torques on a satellite of the Earth as a function of the orbit height h: M_g gravity torque; M_a aerodynamic torque, M_H magnetic torque; M_r solar radiation torque, M_m micrometeorite impact torque.

(A) Gravity Gradient Torque On A Satellite

In this section, we first derive the equation for the G.G.T.; second its order of magnitude is discussed; third, by re-arranging terms, we express the G.G.T. equation in a particular form by grouping the orbit-influenced terms and the attitude-influenced terms separately, so that it can be handily applied in our asymptotic analysis.

1. Equation for Gravity Gradient Torque

with $\bar{R}, \bar{\rho}$ and c as defined in Fig. 3.4.1, let $R = |\bar{R}|$, and further define $\bar{r} = \bar{R} + \bar{\rho}$, and $r = |\bar{r}|$.



C- center of mass

R- position vector

Fig. 3.4.1

The gravity attraction force acting on the mass dm by the earth is:

$$d\bar{F} = -u \; dm \; \bar{r}/r^3 \qquad (3.4.1)$$

where u is called the gravitation constant ($u = 1.407639 \; 10^{16} \; ft^3/sec^2$). The corresponding torque generated by $d\bar{F}$ with respect to center of mass c will be:

$$\begin{aligned} d\bar{L} &= \bar{\rho} \times d\bar{F} \\ &= \frac{-\mu \, dm}{r^3} (\bar{\rho} \times \bar{r}) \\ &= \frac{-\mu \, dm}{r^3} [\bar{\rho} \times (\bar{R} + \bar{\rho})] \\ &= \frac{-\mu \, dm}{r^3} (\bar{\rho} \times \bar{R}) \qquad (3.4.2) \end{aligned}$$

We note that:

$$\begin{aligned} r^2 = \bar{r} \cdot \bar{r} &= (\bar{R} + \bar{\rho}) \cdot (\bar{R} + \bar{\rho}) \\ &= R^2 \left[1 + \frac{2\bar{R} \cdot \bar{\rho}}{R^2} + \frac{\rho^2}{R^2}\right] + \cdots \\ r^{-3} = (r^2)^{-3/2} &= R^{-3} \left[1 - \frac{3\bar{R} \cdot \bar{\rho}}{R^2}\right] + \cdots \qquad (3.4.3) \end{aligned}$$

We have omitted all the terms smaller than $1/R^4$. The total torque acting upon the satellite using (3.4.2) and (3.4.3) will be:

$$\bar{L}_G = \int_m d\bar{L}_G$$

$$= \frac{-\mu}{R^3}\int_m \left[1 - \frac{3\bar{R}\cdot\bar{\rho}}{R^2}\right](\bar{\rho}\times\bar{R})\, dm$$

$$= \frac{-\mu}{R^3}\int_m \bar{\rho}\times\bar{R}\, dm + \frac{\mu}{R^3}\int_m \frac{3(\bar{R}\cdot\bar{\rho})}{R^2}(\bar{\rho}\times\bar{R})\, dm$$

By definition, because c is the center of mass, the first term in the above equation has to be zero, therefore:

$$\bar{L}_G = \frac{3\mu}{R^5}\int_m (\bar{R}\cdot\bar{\rho})(\bar{\rho}\times\bar{R})\, dm \qquad (3.4.4)$$

If L is expressed in body principal axes, let:

$$\bar{R}^b = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix}, \qquad \bar{\rho}^b = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}$$

Then,

$$\bar{L}_G^b = \frac{3\mu}{R^5}\int_m \begin{bmatrix} R_1R_3\rho_1\rho_2 + R_2R_3\rho_2^2 + R_3^2\rho_2\rho_3 \\ -R_1R_2\rho_1\rho_3 - R_2R_2\rho_2\rho_3 - R_3R_2\rho_3^2 \\ R_1^2\rho_1\rho_3 + R_1^2\rho_2\rho_3 + R_3R_1\rho_3^2 \\ -R_1R_3\rho_1^2 - R_2R_3\rho_1\rho_2 - R_3^2\rho_1\rho_3 \\ R_1R_2\rho_1^2 + R_2^2\rho_1\rho_2 + R_2R_3\rho_1\rho_3 \\ -R_1^2\rho_1\rho_2 - R_2R_1\rho_2^2 - R_1R_3\rho_3\rho_2 \end{bmatrix} dm$$

Using the fact that, in body fixed principal axes, all the cross product moments of inertia are zero,

$$\int_m \rho_i\rho_j = 0 \qquad \text{if} \quad i \neq j$$

We have, therefore,

$$\bar{L}_G^b = \frac{3\mu}{R^5}\begin{bmatrix} R_2R_3\,(I_3 - I_y) \\ R_3\,R_1\,(I_x - I_3) \\ R_1\,R_2\,(I_y - I_x) \end{bmatrix} \qquad (3.4.5)$$

In vector notation :

$$\bar{L}_G^b = \frac{3\mu}{R^3}\left[\frac{\bar{R}^b}{R}\times\right] \overset{*}{I}_m \left(\frac{\bar{R}^b}{R}\right) \tag{3.4.6}$$

where

$$\overset{*}{I}_m = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}.$$ is the matrix of moment of inertia

and

$$I_x = \int (\rho_2^2 - \rho_3^2)\, dm$$

$$I_y = \int (\rho_3^2 - \rho_1^2)\, dm$$

$$I_z = \int (\rho_1^2 - \rho_2^2)\, dm$$

(3.4.6) is the equation for the gravity gradient torque; it has a simple format.

2) Order of magnitude considerations

Since G.G.T. is the major disturbing torque for satellite attitude motion, it is important to know its order of magnitude. As a matter of fact, this information - order of magnitude - plays an essential role in our analysis of the dynamics by asymptotic methods.

From orbital dynamics [37], the magnitude of the position vector R, can be expressed in terms of eccentricity e and true anomaly f, that is:

$$R = \frac{a(1-e^2)}{1+e\cos f} \tag{3.4.7}$$

where 'a' is the orbit semi-major axis. The orbit period 'p' is:

$$p = \frac{2\pi}{\omega_{orbit}} = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{3.4.8}$$

Combining (3.4.6), (3.4.7), and (3.4.8) we have:

$$\bar{L}_G^b = \frac{3\ \omega_{orbit}^2\ (1+e\cos f)^3}{(1-e^2)^3}\left(\frac{\bar{R}^b}{R}\times\right)\overset{*}{I}_m\left(\frac{\bar{R}^b}{R}\right)$$

$$= O\left(\omega_{orbit}^2\right) \tag{3.4.9}$$

We can say that the G.G.T. has the order of the square of the orbital frequency, if the eccentricity e is far from one (parabolic if e=1), and if the moment of inertia matrix $\overset{*}{I}_m$ is not approximately an identity matrix (special mass distribution).

3) Re-grouping Terms for G.G.T.

We find that $\bar{R}^b$ - the orbital position vector expressed in the body fixed coordinates - contains both the orbital and the attitude modes. Inspecting the G.G.T. equation (3.4.6), it seems possible to group the orbital and the attitude modes separately.

Since

$$\bar{R}^i = R\begin{pmatrix}\cos f\\ 0\\ -\sin f\end{pmatrix}$$

where 'i' denotes perigee coordinated, we write

$$\bar{R}^b = \overset{*}{C}_{Nb} \overset{*}{C}_{iN} \bar{R}^i$$

$$\bar{L}_G^b = \frac{3\,\omega_{orbit}^2 (1+e\cos f)^2}{(1-e^2)^3} \left[\overset{*}{C}_{ib} \begin{pmatrix} \cos f \\ 0 \\ -\sin f \end{pmatrix} \times \right] \overset{*}{I}_m \overset{*}{C}_{ib} \begin{pmatrix} \cos f \\ 0 \\ -\sin f \end{pmatrix}$$

(3.4.10)

Let us define a linear operator $\overset{*}{OP}(\overset{*}{B})$ on a matrix $\overset{*}{B}$ by:

$$\overset{*}{OP}(\overset{*}{B}) = \begin{bmatrix} -B_{23} & 0 & B_{21} \\ B_{13} & -B_{31} & -B_{11}+B_{33} \\ 0 & B_{21} & -B_{23} \end{bmatrix} \qquad (3.4.11)$$

It is straightforward to show that:

$$\left[\begin{pmatrix} \cos f \\ 0 \\ -\sin f \end{pmatrix} \times \right] \overset{*}{B} \begin{pmatrix} \cos f \\ 0 \\ -\sin f \end{pmatrix} = \overset{*}{OP}(\overset{*}{B}) \begin{pmatrix} \sin^2 f \\ \cos^2 f \\ \sin f \cos f \end{pmatrix} \qquad (3.4.12)$$

Substituting (3.4.12) into (3.4.10) we have:

$$\bar{L}_G^b = \left[\overset{*}{C}_{ib}\, \overset{*}{OP}(\overset{*}{C}_{bi}\, \overset{*}{I}_m\, \overset{*}{C}_{ib}) \right] \cdot \left[\frac{3\,\omega_{orbit}^2 (1+e\cos f)^3}{(1-e^2)^3} \begin{pmatrix} \sin^2 f \\ \cos^2 f \\ \sin f \cos f \end{pmatrix} \right] \qquad (3.4.13)$$

From the above equation it is clear that the first group contains terms influenced by the attitude motion with higher frequency, and the second group is influenced by the orbital motion with relatively lower frequency.

(B) Geomagnetic Torque

A satellite in orbit around the earth interacts with the geomagnetic field and the torque produced by this interaction can be defined as a vector product

$$\bar{L}_M = \bar{V}_M \times \bar{B} \qquad (3.4.14)$$

where $\bar{B}$ is the geomagnetic field and $\bar{V}_M$ is the magnetic moment of the spacecraft. The latter could arise from any current-carrying devices in the satellite payload as well as the eddy currents in the metal structure, which cause undesirable disturbing torques. On the other hand, the vehicle magnetic moment could also be generated purposely by passing an electric current through an onboard coil to create a torque for attitute control.

If the geomagnetic field $\bar{B}$ is modeled as a dipole, it has the form [38].

$$\bar{B} = \frac{\mu_B}{R^5} \left[R^2 \bar{e}_B - 3 \left(\bar{e}_B \cdot \bar{R} \right) \bar{R} \right] \qquad (3.4.15)$$

where $\bar{e}_B$ is a unit vector in the direction of the geomagnetic dipole axis, which inclines about 11.5 degrees from the geophysical polar axis. Vector $\bar{R}$ represents the satellite position vector, μ_B is the magnetic constant of the earth ($\mu_B = 8.1 \times 10^{25}$ gauss-cm^3).

Combining equations (3.4.14) and (3.4.15) and expressing $\bar{L}_M$ in body fixed coordinates, we have

$$\bar{L}_M^b = [\bar{V}_M^b \times] \overset{*}{C}_{ib} \frac{\mu_B}{R^5} [R^2 \bar{e}_B^i - 3(\bar{e}_B^i \cdot \bar{R}^i)\bar{R}^i] \tag{3.4.16}$$

Although neither the geomagnetic field nor the body magnetic moment can be determined precisely in general, modeling both of them as dipoles will be sufficiently accurate for our purpose.

Summary of the Section

1. Gravity gradient torque (G.G.T.) and geomagnetic torque (G.M.T.) are by far the most influential disturbing torques on satellite attitude motion.

2. The basic equation for G.G.T. is:

$$\bar{L}_G^b = \frac{3\mu}{R^3} \left[\frac{\bar{R}}{R} \times\right] \overset{*}{I}_m \left(\frac{\bar{R}}{R}\right) \tag{3.4.6}$$

For G.M.T. is

$$\bar{L}_M^b = (\bar{V}_M^b \times) \overset{*}{C}_{ib} \frac{\mu_B}{R^3} \left[\bar{e}_B^i - 3(\bar{e}_B^i \cdot \bar{R}^i)\frac{\bar{R}^i}{R^2}\right] \tag{3.4.16}$$

3. G.G.T. and G.M.T. have the same order as ω_{orbit}^{2}, if the eccentricity is not too high and if the satellite mass distribution is not too nearly spherical.

4. By re-grouping, we separated terms of attitude frequency and terms of orbital frequency in $\bar{L}_G^b$ and $\bar{L}_M^b$; the results are (3.4.13) and (3.4.16).

3.5 Asymptotic Approach for Attitude Motion With Small Disturbances

(A) Euler's equations

Euler's equations in vector form are ;

$$\overset{*}{I}_m \dot{\bar{\omega}} + [\bar{\omega} \times] \overset{*}{I}_m \bar{\omega} = \varepsilon^2 T_1 + \varepsilon^3 T_2 + \cdots \quad (3.5.1)$$

assuming the initial condition is

$$\bar{\omega}(t_0) = \bar{\omega}_0$$

where $\varepsilon^2 T_1 + \varepsilon^3 T_2 + \cdots$ represent the disturbing torques. The order of magnitude of these disturbing torques are discussed in the previous section. The small parameter ε is defined as the ratio of orbital and attitude frequencies.

Let $\bar{\omega}_N(t)$ be the torque-free Kirchhoff's solution which satisfies the particular initial condition ; that is

$$\overset{*}{I}_m \dot{\bar{\omega}}_N + [\bar{\omega}_N \times] \overset{*}{I}_m \bar{\omega}_N = 0 \quad (3.5.2)$$

$$\text{I. C.} \qquad \bar{\omega}_N(t_0) = \bar{\omega}_0$$

By Encke's approach [37], let

$$\bar{\omega}(t) = \bar{\omega}_N(t) + \varepsilon\, \overline{\delta\omega}(t) \quad (3.5.3)$$

Substituting (3.5.3) into (3.5.1) and subtracting (3.5.2), we have the equation for $\delta\omega(t)$

$$\overset{*}{I}_m \, \delta\dot{\bar{\omega}} + [\bar{\omega}_N \times] \, \overset{*}{I}_m \, \delta\bar{\omega} + [\delta\bar{\omega} \times] \, \overset{*}{I}_m \, \bar{\omega}_N + \varepsilon \, [\delta\bar{\omega} \times] \, \overset{*}{I}_m \, \delta\bar{\omega} = \varepsilon T_1 + \varepsilon^2 T_2 + \cdots \tag{3.5.4}$$

$$\delta\bar{\omega}(t_0) = \bar{0}$$

Note that Encke's perturbational approach is not an approximate method. Because by combining (3.5.4) and (3.5.2) the original equation can be reconstructed. Nevertheless, performing the computation in this perturbational form, one has the advantage of reducing the numerical round-off errors.

For simplifying the notation, a periodic matrix operator $\overset{*}{A}(t)$ with period of T_ω can be defined as :

$$\overset{*}{A}(t) = - \overset{*}{I}{}_m^{-1} \left[(\bar{\omega}_N \times) \, \overset{*}{I}_m - (\overset{*}{I}_m \, \bar{\omega}_N \times) \right] \tag{3.5.5}$$

Eq. (3.5.4) can be re-written as

$$\delta\dot{\bar{\omega}} - \overset{*}{A}(t) \, \delta\bar{\omega} + \varepsilon \, \overset{*}{I}{}_m^{-1} \, (\delta\bar{\omega} \times) \, \overset{*}{I}_m \, \delta\bar{\omega} = \varepsilon \, \overset{*}{I}{}_m^{-1} \, \bar{T}_1 + \varepsilon^2 \, \overset{*}{I}{}_m^{-1} \, T_2 + \cdots \tag{3.5.6}$$

We see that the above equation is a weakly non-linear equation, because the non-linear terms are at least one order smaller than the linear ones.

Further, by Floquet theory , a linear equation with periodic coefficient, such as $\overset{*}{A}(t)$ in (3.5.6), can be reduced to a constant coefficient equation, as follows.

Let matrices $\overset{*}{R}_A$ and $\overset{*}{P}_A^{-1}(t)$ be defined as

$$\overset{*}{R}_A = \frac{1}{T_\omega} \ln \left[\Phi_A (T_\omega, 0) \right]$$

and (3.5.7)

$$\overset{*}{P}_A^{-1}(t) = \Phi_A (t, 0)\, e^{-\overset{*}{R}_A t}$$

where $\overset{*}{\Phi}_A (t, 0)$ is the transition matrix for $\overset{*}{A}(t)$.

It can be proven that [36] ,

(1) $\overset{*}{P}_A^{-1}(t)$ is a periodic matrix

$$\overset{*}{P}_A^{-1} (t + T_\omega) = \overset{*}{P}_A^{-1} (t)$$

(2) $$\overset{*}{P}_A(t)\ \overset{*}{A}(t)\ \overset{*}{P}_A^{-1}(t) + \frac{d\,\overset{*}{P}_A(t)}{dt}\ \overset{*}{P}_A^{-1}(t)$$

$$= \overset{*}{R}_A$$, a constant matrix

Let $\delta\bar{\omega}(t)$ be transformed into $\bar{u}(t)$ by

$$\bar{u} = \overset{*}{P}_A(t)\ \delta\bar{\omega} \tag{3.5.8}$$

Then

$$\begin{aligned}
\dot{\bar{u}} &= \dot{\overset{*}{P}}_A\ \delta\bar{\omega} + \overset{*}{P}_A\ \delta\dot{\bar{\omega}} \\
&= \dot{\overset{*}{P}}_A\ \overset{*}{P}{}_A^{-1}\,\bar{u} + \overset{*}{P}_A \Big[\overset{*}{A}\ \delta\bar{\omega} - \varepsilon\ \overset{*}{I}{}_m^{-1}\,(\delta\bar{\omega}\times)\ \overset{*}{I}_m\,\delta\bar{\omega} \\
&\qquad + \varepsilon\ \overset{*}{I}{}_m^{-1}\ \bar{T}_1 + \varepsilon^2\ \overset{*}{I}{}_m^{-1}\ \bar{T}_2 + \cdots \Big] \\
&= \big(\dot{\overset{*}{P}}_A\ \overset{*}{P}{}_A^{-1} + \overset{*}{P}_A\ \overset{*}{A}\ \overset{*}{P}{}_A^{-1} \big)\ \bar{u} \\
&\quad - \varepsilon\ \overset{*}{P}_A\ \overset{*}{I}{}_m^{-1}\ (\overset{*}{P}{}_A^{-1}\,\bar{u}\times)\ \overset{*}{I}_m\ (\overset{*}{P}{}_A^{-1}\,\bar{u}) \\
&\quad + \varepsilon\ \overset{*}{P}_A\,\overset{*}{I}{}_m^{-1}\ \bar{T}_1 + \varepsilon^2\ \overset{*}{P}_A\ \overset{*}{I}{}_m^{-1}\ \bar{T}_2 + \cdots \\
&= \overset{*}{R}_A\ \bar{u} - \varepsilon\ \overset{*}{P}_A\ \overset{*}{I}{}_m^{-1}\ (\overset{*}{P}{}_A^{-1}\,\bar{u}\times)\ \overset{*}{I}_m (\overset{*}{P}{}_A^{-1}\,\bar{u}) \\
&\quad + \varepsilon\ \overset{*}{P}_A\ \overset{*}{I}{}_m^{-1}\ \bar{T}_1 + \varepsilon^2\,\overset{*}{P}_A\ \overset{*}{I}{}_m^{-1}\ \bar{T}_2 + \cdots
\end{aligned} \tag{3.5.9}$$

These results can be proved by using eq.(3.5.5) and the property (2).

Moreover, in our case, the constant matrix $\overset{*}{R}_A$ is proportional to $1/T_{\omega}$; therefore it can be considered to be of order ε . Also, for simplicity, $\overset{*}{R}_A$ can

be transformed into a diagonal (or Jordan) form

Such as:

$$\overset{*}{M}^{-1}\overset{*}{R}_A\overset{*}{M} = \overset{*}{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{3.5.10}$$

Let
$$\bar{v} = \overset{*}{M}^{-1}\bar{u} \tag{3.5.11}$$

Substitute $\bar{v}$ into (3.5.9); finally we have

$$\begin{aligned}\dot{\bar{v}} = \varepsilon\overset{*}{\Lambda}\bar{v} - \varepsilon\overset{*}{Q}(\overset{*}{I}_m^{-1}\overset{*}{Q}^{-1}\bar{v}\times)\overset{*}{Q}^{-1}\dot{\bar{v}} \\ + \overset{*}{Q}(\varepsilon\bar{T}_1 + \varepsilon^2\bar{T}_2 + \cdots)\end{aligned} \tag{3.5.12}$$

where

$$\overset{*}{Q} = \overset{*}{M}^{-1}\overset{*}{P}_A\overset{*}{I}_m^{-1} \tag{3.5.13}$$

MTS Asymptotic Solution

For solving (3.5.12), we apply the linear multiple time scales asymptotic technique [3]. We first expand the time domain t into a multi-dimensional space,

$$t \rightarrow (\tau_0, \tau_1, \tau_2, \cdots)$$

and τ_0, τ_1, ...etc. are defined as ;

$$\tau_0 = t ; \quad \tau_1 = \varepsilon t ; \quad \tau_2 = \varepsilon^2 t ; \cdots \text{etc,}$$

The time derivatives in the new dimensions are transformed according to

$$\begin{aligned} \frac{d}{dt} &\rightarrow \frac{\partial}{\partial \tau_0} + \frac{\partial}{\partial \tau_1}\frac{d\tau_1}{dt} + \frac{\partial}{\partial \tau_2}\frac{d\tau_2}{dt} + \cdots \\ &= \frac{\partial}{\partial \tau_0} + \varepsilon \frac{\partial}{\partial \tau_1} + \varepsilon^2 \frac{\partial}{\partial \tau_2} + \cdots \end{aligned} \tag{3.5.14}$$

Also we assume that the dependent variable $\bar{v}(t)$ can be expanded into an asymptotic series of ε :

$$\bar{v}(t) \simeq \bar{v}_0(\tau_0, \tau_1 \cdots) + \varepsilon\, \bar{v}_1(\tau_0, \tau_1) + \cdots \tag{3.5.15}$$

Substituting (3.5.14) and (3.5.15) into (3.5.12) we have :

$$\frac{\partial \bar{U}_0}{\partial \tau_0} + \varepsilon\left(\frac{\partial \bar{U}_0}{\partial \tau_1} + \frac{\partial \bar{U}_1}{\partial \tau_0}\right) + \varepsilon^2\left(\frac{\partial \bar{U}_0}{\partial \tau_2} + \frac{\partial \bar{U}_1}{\partial \tau_1} + \frac{\partial \bar{U}_2}{\partial \tau_0}\right) + \cdots$$

$$= \varepsilon\left[\overset{*}{\Lambda}\,\bar{U}_0 - \overset{*}{Q}\,(\overset{*}{I}{}_m^{-1}\,\overset{*}{Q}{}^{-1}\,\bar{U}_0 \times)\,\overset{*}{Q}{}^{-1}\bar{U}_0 + \overset{*}{Q}\,\bar{T}_1\right]$$

$$+ \varepsilon^2\left[\overset{*}{\Lambda}\,\bar{U}_1 - \overset{*}{Q}\,(\overset{*}{I}{}_m^{-1}\,\overset{*}{Q}{}^{-1}\bar{U}_1 \times)\,\overset{*}{Q}{}^{-1}\bar{U}_0\right.$$

$$\left. - \overset{*}{Q}\,(\overset{*}{I}{}_m^{-1}\,\overset{*}{Q}{}^{-1}\,\bar{U}_0 \times)\,\overset{*}{Q}{}^{-1}\bar{U}_1 + \overset{*}{Q}\,\bar{T}_2\right] + \cdots \quad (3.5.16)$$

By equating the coefficients of like powers of ε on both sides of (3.5.16), we have:

Coefficient of ε^0 :

$$\frac{\partial \bar{U}_0}{\partial \tau_0} = 0 \quad (3.5.17)$$

Coefficient of ε^1 :

$$\frac{\partial \bar{U}_1}{\partial \tau_0} = -\frac{\partial \bar{U}_0}{\partial \tau_1} + \overset{*}{\Lambda}\,\bar{U}_0 - \overset{*}{Q}\left[\overset{*}{I}{}_m^{-1}\,\overset{*}{Q}{}^{-1}\,\bar{U}_0 \times\right]\overset{*}{Q}{}^{-1}\,\bar{U}_0$$

$$+ \overset{*}{Q}\,\bar{T}_1 \quad (3.5.18)$$

Coefficient of ε^2:

$$\frac{\partial \bar{V}_2}{\partial \tau_0} = \frac{\partial \bar{V}_1}{\partial \tau_1} + \frac{\partial \bar{V}_0}{\partial \tau_2} + \Lambda \bar{V}_1 - \overset{*}{Q}\Big[\big(\overset{*}{I}{}_m^{-1} \overset{*}{Q}{}^{-1} \bar{V}_0 \times \big) \overset{*}{Q}{}^{-1} \bar{V}_1$$

$$+ \big(\overset{*}{I}{}_m^{-1} \overset{*}{Q}{}^{-1} \bar{V}_1 \times \big) \overset{*}{Q}{}^{-1} \bar{V}_0 \Big] + \overset{*}{Q} \bar{T}_2 \quad (3.5.19)$$

$\vdots$

etc

Solving the partial differential equation (3.5.17), we have

$$\bar{V}_0 = \bar{V}_0 (\tau_1 , \tau_2 \cdots) \quad (3.5.20)$$

where $\bar{V}_0$ is not a function of τ_0, and the initial condition is

$$\bar{V}_0 (0) = 0$$

and $\bar{V}_0 (\tau_1 ; \tau_2)$ is yet to be determined. Note, (3.5.20) implies that the zeroth order solution for Euler's equation with small disturbances is the Kirchhoff's solution.

Substituting (3.5.20) into (3.5.18), we have

$$\frac{\partial \bar{V}_1}{\partial \tau_0} = - \frac{\partial \bar{V}_0}{\partial \tau_1} + \overset{*}{\Lambda} \bar{V}_0 - \overset{*}{Q} \big(\overset{*}{I}{}_m^{-1} \overset{*}{Q}{}^{-1} \bar{V}_0 \times \big) \overset{*}{Q}{}^{-1} \bar{V}_0$$

$$+ \overset{*}{Q} \bar{T}_1 \quad (3.5.21)$$

This equation with the initial condition $\bar{v}_1(0) = 0$ can be solved as,

$$\bar{v}_1 = \left(-\frac{\partial \bar{v}_0}{\partial \tau_1} + \overset{*}{\Lambda}\bar{v}_0\right)\tau_0 - \int_0^{\tau_0} \overset{*}{Q}(\overset{*}{I}{}_m^{-1}\overset{*}{Q}{}^{-1}\bar{v}_0 \times)\overset{*}{Q}{}^{-1}\bar{v}_0\, d\tau_0$$

$$+ \int_0^{\tau_0} \overset{*}{Q}\,\bar{T}_1\, d\tau_0 \qquad (3.5.22)$$

The term $\overset{*}{Q}(\overset{*}{I}{}_m^{-1}\overset{*}{Q}{}^{-1}\bar{v}_0 \times)\overset{*}{Q}{}^{-1}\bar{v}_0$ in the above equation can be written, as shown in Appendix A, as follows:

$$\overset{*}{Q}(\overset{*}{I}{}_m^{-1}\overset{*}{Q}{}^{-1}\bar{v}_0 \times)\overset{*}{Q}{}^{-1}\bar{v}_0$$

$$= \overset{*}{F}_1(\tau_0)\, v_{0_1}\bar{v}_0 + \overset{*}{F}_2(\tau_0)\, v_{0_2}\bar{v}_0 + \overset{*}{F}_3(\tau_0)\, v_{0_3}\bar{v}_0$$

$$= \sum_{i=1}^{3} \overset{*}{F}_i(\tau_0)\, v_{0_i}\bar{v}_0 \qquad (3.5.23)$$

where $\overset{*}{F}_1$, $\overset{*}{F}_2$ and $\overset{*}{F}_3$ are periodic matrices with period of T_ω, and v_{0_i}, $i = 1, 2, 3$ are three components of the vector $\bar{v}_0$. If we expand $\overset{*}{F}_1$, $\overset{*}{F}_2$ and $\overset{*}{F}_3$ into Fourier series

we have ;

$$\overset{*}{Q} \, (\overset{*}{I}_m^{-1} \overset{*}{Q}^{-1} \bar{U}_0 \times) \, \overset{*}{Q}^{-1} \bar{U}_0$$

$$\approx \sum_{i=1}^{3} \sum_{j=0}^{n} \left[\overset{*}{E}_{ij} \sin\left(\frac{2\pi j \tau_0}{T_\omega}\right) + \overset{*}{F}_{ij} \cos\left(\frac{2\pi j \tau_0}{T_\omega}\right)\right] U_{0_i} \bar{U}_0 \qquad (3.5.25)$$

where $\overset{*}{E}_{ij}$ and $\overset{*}{F}_{ij}$ are constant matrices. Also, in case that the external torque T_1 is not an explicit function of time, then $\overset{*}{Q} T_1$ can be transformed into a particular form as,

$$\overset{*}{Q} \bar{T}_1 = \bar{G}_1 (\tau_1, \tau_2 \cdots) + \overset{*}{Q}_p(\tau_0) \, \bar{G}_2 (\tau_1, \tau_2 \cdots) \qquad (3.5.26)$$

where $\bar{G}_1$, $\bar{G}_2$ are functions of slow variables $\tau_1, \tau_2, \cdots$ only, and $\overset{*}{Q}_p(\tau_0)$ is a fast time-varying function. The expressions of (3.5.26) for gravity gradient torque and geomagnetic torque are given in the following sections.

Substituting (3.5.25) and (3.5.26) into (3.5.22),

$$\bar{U}_1 = \left[-\frac{\partial \bar{U}_0}{\partial \tau_1} + \overset{*}{\Lambda} \bar{U}_0 - \sum_{i=1}^{3} \overset{*}{F}_{i0} U_{0_i} \bar{U}_0 + \bar{G}_1(\tau_1) \right] \tau_0$$

$$- \int_0^{\tau_0} \sum_{i=1}^{3} \sum_{j=1}^{n} \left[\overset{*}{E}_{ij} \sin\left(\frac{2\pi j \tau_0}{T_\omega}\right) + \overset{*}{F}_{ij} \cos\left(\frac{2\pi j \tau_0}{T_\omega}\right) \cdot \right.$$

$$v_{0_i} \bar{v}_0 \, d\tau_0 + \int_0^{\tau_0} \overset{*}{Q}_p(\tau_0) \, \bar{G}_2(\tau_1) \, d\tau_0 \qquad (3.5.27)$$

For achieving a uniformly valid approximate solution, it is required that $\|\bar{v}_1(\tau_0,\tau_1)\| / \|\bar{v}_0(\tau_0,\tau_1)\|$ be bounded uniformly for all time . Applying this condition to (3.5.27), it requires the first bracket to the right of the equal sign to be zero since, otherwise, it will linearly increase with τ_0. This leads to the equation:

$$\frac{\partial \bar{v}_0}{\partial \tau_1} = \overset{*}{\Lambda} \, \bar{v}_0 - \sum_{i=1}^{3} \overset{*}{F}_{i0} \, v_{0_i} \, \bar{v}_0 + \bar{G}_1(\tau_1) \qquad (3.5.28)$$

with I. C.

$$\bar{v}_0(0) = 0$$

Also eq. (3.5.27) becomes

$$\bar{v}_1(\tau_0,\tau_1) \approx - \int_0^{\tau_0} \sum_{i=1}^{3} \sum_{j=1}^{n} \Big[\overset{*}{E}_{ij} \sin\left(\frac{2\pi j \sigma}{T_\omega}\right) + \overset{*}{F}_{ij} \cos\left(\frac{2\pi j \sigma}{T_\omega}\right)\Big] v_{0_i} \, \bar{v}_0 \, d\sigma + \int_0^{\tau_0} \overset{*}{Q}_p(\sigma) \, \bar{G}_2(\tau_1) \, d\sigma \qquad (3.5.29)$$

Eqs. (3.5.28) and (3.5.29) yield the first order multiple time scales asymptotic solution to the equation of $\bar{v}(t)$, (3.5.12). Once $\bar{v}(t)$ is obtained, the asymptotic solution of Euler's equation (3.5.1) can be constructed as ;

$$\begin{aligned} \bar{\omega}(t) &= \bar{\omega}_N + \delta\bar{\omega} \\ &= \bar{\omega}_N(t) + \varepsilon \overset{*}{P}_A^{-1} \overset{*}{M} (\bar{v}_0 + \varepsilon v_1) \end{aligned} \tag{3.5.30}$$

The matrices $\overset{*}{P}_A^{-1}$ and $\overset{*}{M}$ are given by (3.5.7) and (3.5.10).

The above approach gives us an alternative way of evaluating a satellite angular velocity $\bar{\omega}(t)$ instead of direct integration of Euler's equation. By this new approach we integrate (3.5.28) directly and evaluate (3.5.29) analytically. Since (3.5.28) is in terms of $\tau_1(t,\varepsilon)$, it allows us to use a large integration time step, thus saving computer time. Also, we note that (3.5.29) describes the oscillatory motion and (3.5.28) describes the secular motion of the satellite angular velocity.

(B) Solution For Euler Symmetric Parameters

Euler symmetric parameters are kinematically related to the angular velocity by a linear differential equation (3.2.9)

$$\dot{\bar{\beta}} = \frac{1}{2}\,[\overset{*}{\omega}]\,\bar{\beta} \tag{3.5.31}$$

From (3.5.30)

$$\begin{aligned}\bar{\omega} &= \bar{\omega}_N + \delta\bar{\omega} \\ &= \bar{\omega}_N + \varepsilon\,\overset{*}{P}_A^{-1}\,\overset{*}{M}\,(\bar{v}_0 + \varepsilon\,\bar{v}_1)\end{aligned}$$

Hence

$$\dot{\bar{\beta}} = \frac{1}{2}\left\{[\overset{*}{\omega}_N] + \varepsilon\,[\overset{*}{\delta\omega}]\right\}\bar{\beta} \tag{3.5.32}$$

Again by linear multiple time scales method

$$t \rightarrow [\tau_0, \tau_1, \cdots]$$

$$\tau_0 = t$$

$$\tau_1 = \varepsilon t$$

$$\vdots$$

etc

we have

$$\frac{d}{dt} = \frac{\partial}{\partial\tau_0} + \frac{\varepsilon\,\partial}{\partial\tau_1} + \cdots\cdots \tag{3.5.33}$$

Assume that

$$\bar{\beta}(t) = \bar{\beta}_0(\tau_0, \tau_1 \cdots) + \varepsilon\, \bar{\beta}_1(\tau_0, \tau_1 \cdots) + \cdots \qquad (3.5.34)$$

Substituting (3.5.33) and (3.5.34) into (3.5.32), and arranging terms in the power of ε , we have :

Coefficient of ε^0 :

$$\frac{\partial \bar{\beta}_0}{\partial \tau_0} = \frac{1}{2}\,[\overset{*}{\omega}_N]\,\bar{\beta}_0 \qquad (3.5.35)$$

Coefficient of ε^1 :

$$\frac{\partial \bar{\beta}_1}{\partial \tau_0} = \frac{1}{2}\,[\overset{*}{\omega}_N]\,\bar{\beta}_1 - \frac{\partial \bar{\beta}_0}{\partial \tau_1} + \frac{1}{2}\,[\delta \overset{*}{\omega}]\,\bar{\beta}_0 \qquad (3.5.36)$$

Coefficient of ε^2 :

$$\frac{\partial \bar{\beta}_2}{\partial \tau_0} = \frac{1}{2}\,[\overset{*}{\omega}_N]\,\bar{\beta}_2 - \frac{\partial \bar{\beta}_0}{\partial \tau_2} - \frac{\partial \bar{\beta}_1}{\partial \tau_1} - \frac{1}{2}[\delta \overset{*}{\omega}]\,\bar{\beta}_1 \qquad (3.5.37)$$

$\vdots$

etc

Let $\Phi_\beta(t, t_0)$ be the transition matrix for $\frac{1}{2}[\overset{*}{\omega}_N]$; that is

$$\frac{d}{dt}\overset{*}{\Phi}_\beta = \frac{1}{2}\,[\overset{*}{\omega}_N]\,\overset{*}{\Phi}_\beta$$

The expression for $\Phi_\beta(t, t_0)$ is given by (3.3.29), that is from the solution of β_i by Morton's

approach. Similarly, $\Phi_\beta(t,t_0)$ can be also achieved by using the Floquet theorem, although the latter one is more numerically oriented and requires several transformations.

The solution for the ε^0 order equation (3.5.35) is

$$\bar{\beta}_0(\tau_0,\tau_1) = \overset{*}{\Phi}_\beta(\tau_0,0)\,\bar{\beta}_{0N}(\tau_1,\tau_2\cdots) \tag{3.5.38}$$

with I.C.

$$\bar{\beta}_{0N}(0) = \bar{\beta}(0)$$

and $\bar{\beta}_{0N}(\tau_1,\tau_2\cdots)$ is yet to be determined.

Substituting eq. (3.5.38) into (3.5.36), we have

$$\frac{\partial\bar{\beta}_1}{\partial\tau_0} = \frac{1}{2}[\overset{*}{\omega}_N]\,\bar{\beta}_1 - \overset{*}{\Phi}_\beta\frac{\partial\bar{\beta}_{0N}}{\partial\tau_1} + \frac{1}{2}[\delta\overset{*}{\omega}]\,\overset{*}{\Phi}_\beta\,\bar{\beta}_{0N} \tag{3.5.39}$$

The solution to the above equation is:

$$\begin{aligned}\bar{\beta}_1 &= \overset{*}{\Phi}_\beta(\tau_0,0)\int_0^{\tau_0}\Big[-\frac{\partial\bar{\beta}_{0N}}{\partial\tau_1} + \frac{1}{2}\,\overset{*}{\Phi}{}_\beta^{-1}(\sigma,0)[\delta\overset{*}{\omega}]\,\overset{*}{\Phi}_\beta(\sigma,0)\cdot\bar{\beta}_{0N}\Big]d\sigma \\ &= \overset{*}{\Phi}_\beta(\tau_0,0)\Big[-\frac{\partial\bar{\beta}_{0N}}{\partial\tau_1}\tau_0 + \frac{1}{2}\int_0^{\tau_0}\Phi_\beta^{-1}(\sigma,0)(\delta\overset{*}{\omega})\,\Phi_\beta(\sigma,0)\,\bar{\beta}_{0N}\,d\sigma\Big]\end{aligned} \tag{3.5.40}$$

where $\delta\bar{\omega}(\tau_0,\tau_1)$, given by (3.5.30), is a function

of τ_0 and τ_1 , and $\Phi_\beta(\tau_0, o)$ is a function of τ_0 only. By re-grouping terms it is possible to write $\overset{*}{\Phi}{}_\beta^{-1} [\delta \overset{*}{\omega}] \overset{*}{\Phi}_\beta \bar{\beta}_{oN}$ as follows (appendix B):

$$\overset{*}{\Phi}{}_\beta^{-1} \; [\delta \overset{*}{\omega}] \; \overset{*}{\Phi}_\beta = \overset{*}{R}_1(\tau_1) + \overset{*}{P}_B(\tau_0) \overset{*}{R}_2(\tau_1) \tag{3.5.41}$$

Substituting (3.5.41) into (3.5.40),

$$\bar{\beta}_1 = \overset{*}{\Phi}(\tau_0, o) \left[-\frac{\partial \bar{\beta}_{oN}}{\partial \tau_1} \tau_0 + \frac{1}{2} \overset{*}{R}_1(\tau_1) \bar{\beta}_{oN}(\tau_1) \tau_0 \right.$$
$$\left. + \frac{1}{2} \int_o^{\tau_0} \overset{*}{P}_B(\sigma) \overset{*}{R}_2(\tau_1) \bar{\beta}_{oN}(\tau_1) \, d\sigma \right] \tag{3.5.42}$$

In order to have $\|\bar{\beta}_1\| / \|\bar{\beta}_0\|$ bounded in τ_0, it is necessary that $\bar{\beta}_1$ should not increase with time faster than $\bar{\beta}_0$. Therefore in eq. (3.5.42), those terms which increase linearly with τ_0 have to be set to zero. That is

$$\frac{\partial \bar{\beta}_{oN}}{\partial \tau_1} + \frac{1}{2} \overset{*}{R}_1(\tau_1) \bar{\beta}_{oN}(\tau_1) = o \tag{3.5.43}$$

and (3.5.42) becomes

$$\bar{\beta}_1 = \overset{*}{\Phi}_\beta(\tau_0, o) \left[\frac{1}{2} \int_0^{\tau_0} \overset{*}{P}_B(\sigma) \overset{*}{R}_2(\tau_1) \bar{\beta}_{oN}(\tau_1) \, d\sigma \right] \tag{3.5.44}$$

Equations (3.5.43) and (3.5.44) give an asymptotic solution to the equation of Euler symmetric parameters.

$$\bar{\beta}(\tau_0, \tau_1) = \overset{*}{\Phi}_\beta(\tau_0, 0)\, \bar{\beta}_{0N}(\tau_1) + \varepsilon\, \bar{\beta}_1(\tau_0, \tau_1) + \cdots \tag{3.5.45}$$

where $\bar{\beta}_{0N}(\tau_1)$ from (3.5.43) gives the secular variation of the perturbational motion and $\bar{\beta}_1(\tau_0)$ gives the non-biased oscillatory motions.

Summary of the Section

(A) The rigid body satellite attitude dynamics are described by:

$$\overset{*}{I}_m \bar{\dot{\omega}} + [\omega\times]\, \overset{*}{I}_m \bar{\omega} = \varepsilon^2 T + \cdots \tag{3.5.1}$$

$$\overset{\circ}{\bar{\beta}} = \frac{1}{2}\, [\overset{*}{\omega}]\, \bar{\beta} \tag{3.5.31}$$

which can be replaced by the asymptotic approximate formulation :

$$\frac{\partial \bar{U}_0}{\partial \tau_1} = \overset{*}{\Lambda}\, \bar{U}_0 - \sum_{i=1}^{3} \overset{*}{F}_{i0}\, U_{0_i}\, \bar{U}_0 + \bar{G}_1(\tau_1) \tag{3.5.28}$$

$$\bar{U}_1 = -\int_0^{\tau_0} \sum_{i=1}^{3} \sum_{j=1}^{n} \left[\overset{*}{E}_{ij} \sin\left(\frac{2\pi j \sigma}{T_\omega}\right) + \overset{*}{F}_{ij} \cos\left(\frac{2\pi j \sigma}{T_\omega}\right) \right] U_{0_i} \bar{U}_0$$

$$\cdot\, d\sigma + \int_0^{\tau_0} \overset{*}{Q}_P(\sigma)\, \bar{G}_2(\tau_1)\, d\sigma \tag{3.5.29}$$

$$\omega(t) = \bar{\omega}_N(t) + \varepsilon \overset{*}{P}_A^{-1} \overset{*}{M} \left(\bar{\upsilon}_0 + \varepsilon \bar{\upsilon}_1 \right) \tag{3.5.30}$$

and

$$\begin{cases} \dfrac{\partial \bar{\beta}_{cN}}{\partial \tau_1} + \dfrac{1}{2} \overset{*}{R}_1(\tau_1)\, \bar{\beta}_{cN} = 0 & (3.5.43) \\[2ex] \bar{\beta}_1 = \overset{*}{\Phi}_\beta(\tau_0, 0) \cdot \dfrac{1}{2} \displaystyle\int_0^{\tau_0} \overset{*}{P}_\beta(\sigma) \overset{*}{R}_2(\tau_1)\, \bar{\beta}_{cN}(\tau_1)\, d\sigma & \end{cases}$$

(3.5.44)

$$\beta(t) = \Phi_\beta(t, 0)\, \bar{\beta}_{oN}(t) + \varepsilon \beta_1(t)$$

(3.5.45)

(B) The secular eqs. (3.5.28) and (3.5.43) have to be integrated in $\tau_1(t, \varepsilon)$, or equivalently, in a very large time step in t. The oscillatory equations (3.5.29) and (3.5.44) can be analytically calculated if the external torques $\bar{T}_1$ are not explicit functions of time.

3.6 Attitude Motion With Gravity Gradient Torque

The gravity gradient torque $\bar{L}_G^b$, given by (3.4.13), is :

$$\bar{L}_G^b = \left[\overset{*}{C}_{ib}\ \overset{*}{\mathrm{op}} \left(\overset{*}{C}_{bi}\ \overset{*}{I}_m\ \overset{*}{C}_{ib} \right) \right] \cdot \left[\frac{3\, \omega_{orbit}^2 (1 + e\cos f)^3}{(1 - e^2)^3} \cdot \right.$$

$$\left. \cdot \begin{bmatrix} \sin^2 f \\ \cos^2 f \\ \sin f \cos f \end{bmatrix} \right] \tag{3.6.1}$$

where e, f and ω_{orbit} are the satellite orbit eccentricity, true anomaly and averaged orbital angular velocity, respectively. $\overset{*}{Op}$ is a linear operator given by eq. (3.4.11) .

From (3.3.30)

$$\overset{*}{C}_{nb} = \overset{*}{C}_1{}' \cos(2Rt) + \overset{*}{C}_2{}' \sin(2Rt) + \overset{*}{C}_3{}' \tag{3.6.2}$$

and

$$\begin{aligned}\overset{*}{C}_{ib} &= \overset{*}{C}_{nb} \overset{*}{C}_{in} \\ &= \overset{*}{C}_1{}' \overset{*}{C}_{in} \cos(2Rt) + \overset{*}{C}_2{}' \overset{*}{C}_{in} \sin(2Rt) + \overset{*}{C}_3{}' \overset{*}{C}_{in} \\ &= \overset{*}{C}_1 \cos(2Rt) + \overset{*}{C}_2 \sin(2Rt) + \overset{*}{C}_3\end{aligned} \tag{3.6.3}$$

where $\overset{*}{C}_1$, $\overset{*}{C}_2$ and $\overset{*}{C}_3$ are three periodic matrices with period of T_ω .

Substituting (3.6.3) into (3.6.1)

$$\begin{aligned}\overset{*}{Q}\,\overline{L}_G^{\,b} = \overset{*}{Q}\,(\overset{*}{C}_1 \cos(2Rt) + \overset{*}{C}_2 \sin(2Rt) + \overset{*}{C}_3) \\ \cdot \Big\{ \overset{*}{Op}\,(\overset{*}{C}_1{}^T \overset{*}{I}_m \overset{*}{C}_1) \cos^2 2Rt \\ + \overset{*}{Op}\,(\overset{*}{C}_2{}^T \overset{*}{I}_m \overset{*}{C}_2) \sin^2 2Rt + \overset{*}{Op}\,(\overset{*}{C}_3{}^T \overset{*}{I}_m \overset{*}{C}_3) \\ + \overset{*}{Op}\,(\overset{*}{C}_1{}^T \overset{*}{I}_m \overset{*}{C}_2 + \overset{*}{C}_2{}^T \overset{*}{I}_m \overset{*}{C}_1) \cos 2Rt \sin 2Rt \\ + \overset{*}{Op}\,(\overset{*}{C}_1{}^T \overset{*}{I}_m \overset{*}{C}_3 + \overset{*}{C}_3{}^T \overset{*}{I}_m \overset{*}{C}_1) \cos 2Rt\end{aligned}$$

$$+ \overset{*}{Op}(\overset{*}{C}_2^T \overset{*}{I}_m \overset{*}{C}_2 + \overset{*}{C}_3^T \overset{*}{I}_m \overset{*}{C}_2) \sin 2Rt \; \frac{3\,\omega^2_{orbit}(1+e\cos f)}{(1-e^2)^3}$$

$$\cdot \begin{bmatrix} \sin^2 f \\ \cos^2 f \\ \sin f \cos f \end{bmatrix}$$

or

$$\overset{*}{Q}\,\bar{L}_G^b = \Big\{ \overset{*}{S}_1(t) + \overset{*}{S}_2(t)\sin(2Rt) + \overset{*}{S}_3(t)\cos(2Rt)$$

$$+ \overset{*}{S}_4(t)\sin(4Rt) + \overset{*}{S}_5(t)\cos(4Rt)$$

$$+ \overset{*}{S}_6(t)\sin(6Rt) + \overset{*}{S}_7(t)\cos(6Rt)\Big\}$$

$$\cdot \left\{ \frac{3\,\omega^2_{orbit}(1+e\cos f)^3}{(1-e^2)^3} \begin{pmatrix} \sin^2 f \\ \cos^2 f \\ \sin f \cos f \end{pmatrix} \right\} \quad (3.6.4)$$

where

$$\overset{*}{S}_1 = \overset{*}{Q}\,\overset{*}{C}_3\,\overset{*}{Op}\left(\frac{\overset{*}{C}_1^T \overset{*}{I}_m \overset{*}{C}_1}{2} + \frac{\overset{*}{C}_2^T \overset{*}{I}_m \overset{*}{C}_2}{2} + \overset{*}{C}_3^T \overset{*}{I}_m \overset{*}{C}_3\right)$$

$$+ \frac{1}{2}\overset{*}{Q}\,\overset{*}{C}_2\,\overset{*}{Op}(\overset{*}{C}_2^T \overset{*}{I}_m \overset{*}{C}_3 + \overset{*}{C}_3^T \overset{*}{I}_m \overset{*}{C}_2)$$

$$+ \frac{1}{2}\overset{*}{Q}\,\overset{*}{C}_1\,\overset{*}{Op}(\overset{*}{C}_1^T \overset{*}{I}_m \overset{*}{C}_3 + \overset{*}{C}_3^T \overset{*}{I}_m \overset{*}{C}_1)$$

$$\overset{*}{S}_2 = \;\ldots\ldots$$

etc.

Because $\overset{*}{S}_i$, $i = 1, 2, \cdots 7$ are periodic matrices with period T_ω , they can be expanded into Fourier series with period of T_ω . That is

$$\overset{*}{S}_i = \sum_{j=0}^{n} \left[\overset{*}{M}_{ij} \cos\left(\frac{2\pi j t}{T_\omega}\right) + \overset{*}{N}_{ij} \sin\left(\frac{2\pi j t}{T_\omega}\right) \right] \tag{3.6.5}$$

With that, $\overset{*}{Q}\, \overline{L}_G^b$ can be expressed in the form:

$$\overset{*}{Q}\, \overline{L}_G^b = \overline{G}_1(\tau_1) + \overset{*}{Q}_p(\tau_0)\, \overline{G}_2(\tau_1) \tag{3.6.6}$$

$$\overline{G}_2(\tau_1) = \frac{3\, \omega_{orbit}^2 \cdot (1+\cos f)^3}{(1-e^2)^3} \begin{bmatrix} \sin^2 f \\ \cos^2 f \\ \sin f \cos f \end{bmatrix}$$

$$\overline{G}_1(\tau_1) = \overset{*}{M}_{10}\; \overline{G}(\tau_1) \tag{3.6.7}$$

(3.6.8)

$$\overset{*}{Q}_p(\tau_0) \approx \sum_{j=1}^{n} \left[\overset{*}{M}_{1j} \cos\left(\frac{2\pi j t}{T_\omega}\right) + \overset{*}{N}_{1j} \sin\left(\frac{2\pi j t}{T_\omega}\right) \right]$$
$$+ \sum_{j=0}^{n} \left[\overset{*}{M}_{2j} \cos\left(\frac{2\pi j t}{T_\omega}\right) + \overset{*}{N}_{2j} \sin\left(\frac{2\pi j t}{T_\omega}\right) \right] \sin 2Rt$$
$$+ \sum_{j=0}^{n} \left[\overset{*}{M}_{3j} \cos\left(\frac{2\pi j t}{T_\omega}\right) + \overset{*}{N}_{3j} \sin\left(\frac{2\pi j t}{T_\omega}\right) \right] \cos 2Rt$$

$$+ \cdots$$

$$+ \sum_{j=0}^{n} \left[\overset{*}{M}_{7j} \cos\left(\frac{2\pi j t}{T_\omega}\right) + \overset{*}{N}_{7j} \sin\left(\frac{2\pi j t}{T_\omega}\right)\right] \cos 6Rt \tag{3.6.9}$$

and $\overset{*}{Q}_p(\tau_0)\overline{G}_2(\tau_1)$ can be analytically integrated.

<u>Resonance</u>

In special cases, if the tumbling frequency $\frac{2\pi}{T_\omega}$ and the polhode frequency R are low order commensurable, that is, there exist two small integers n and m such that

$$n\left(\frac{2\pi}{T_\omega}\right) - mR = 0 \tag{3.6.10}$$

then resonance will occur. For handling resonant situations, for example, if $\frac{2\pi}{T_\omega} - 2R = 0$, then in equation (3.6.9) of $\overset{*}{Q}_p(\tau_0)$, terms such as $\sin\frac{2\pi t}{T_\omega} \sin 2Rt$, or $\cos\frac{2\pi t}{T_\omega} \cos 2Rt$ etc. will produce constants (i.c. $\sin\frac{2\pi t}{T_\omega} \sin 2Rt = \sin^2(2Rt)$ $= \frac{1}{2} - \frac{1}{2}\cos 4Rt$, $Const. = \frac{1}{2}$). These constants should be multiplied by $\overline{G}_2(\tau_1)$ and grouped into $\overline{G}_1(\tau_1)$ in (3.6.6). Then our theory with the same formulation enables us to predict the attitude in the resonant case as well.

A Numerical Example

A rigid body asymmetric satellite in an elliptic orbit is simulated. Here, we assume that the satellite attitude motion is influenced by the earth gravity gradient torque only.

The numerical values used in this example are the following:

Satellite moment of inertia:

$I_x = 39.4$ slug-ft^2

$I_y = 33.3$ slug-ft^2

$I_z = 10.3$ slug-ft^2

Orbit parameters of the satellite:

eccentricity $e = 0.16$

inclination $i = 0$

orbital period = 10,000 sec

Initial conditions are:

$\omega_x = 0.0246$ rad/sec

$\omega_y = 0$ rad/sec

$\omega_z = 0$ rad/sec

Satellite starts at orbit perigee and its initial orientation parameters are

$\beta_0 = 0.7071$

$\beta_1 = 0$

$\beta_2 = 0$

$\beta_3 = 0.7071$

In this case, the small parameter ε is about

$$\varepsilon = \frac{\omega_{orbit}}{\omega_{attitude}} \approx 0.03$$

With the above initial conditions, the satellite attitude dynamics are first directly integrated using a fourth order Runge-Kutta method with a small integration time step size of 10 sec for a total interval of 8000 sec. This result is considered to be extremely accurate and referred to as the reference case from here on. The other simulations are then compared to this reference case for checking their accuracy. A number of runs have been tried, both by asymptotic approach and direct integration, with different time step sizes. They are summarized in Table 3.5.1. The errors in each case - i.e., the differences between each simulation and the reference case, - are plotted in terms of time and given by Fig. 3.6.1 through Fig. 3.6.12. Fig. 3.6.13 is a plot of the maximum numerical errors as functions of the step size. From this plot, we see that with direct integration the step size ΔT should be no greater than 25 sec. On the other hand, for asymptotic simulation, the step size can be as large as 500 sec although the first order asymptotic

approximation errors approach zero as $\varepsilon \to 0$ but not as $\Delta t \to 0$. Fig. 3.6.14 is a plot of the required computer time in terms of the step size $\Delta\tau$. We note that for extrapolating a single step, the asymptotic approach requires about double the computer time using direct simulation. However, since the former allows use of a large time step, overall, this new approach will have a significant numerical advantage over direct simulation. In our particular case, the saving is of order 10. Although, in this comparison, we did not include the computer time required for initializing an asymptotic approach by calculating Kirchhoff's solution and some Fourier series expansions etc., we argue that this fixed amount of computer time required for initializing (about 40 sec for the above example) will become only a small fraction of the total, if the prediction task is long. For example, with the above data, if we predict the satellite attitude motion for an interval of three days, the direct integration with a step size $\Delta\tau = 20$ sec requires 1700 sec of computer time, while the new approach with $\Delta\tau = 500$ sec needs about 170 sec plus the initialization of 40 sec.

3.7 Satellite Attitude Motion With Geomagnetic Torque

The geomagnetic torque acting on a satellite can be written as, (3.4.16)

$$\bar{L}_M^b = (\bar{V}_M^b \times)\ \overset{*}{C}_{ib} \frac{\mu_B}{R^5}\left[R^2 \bar{e}_B^i - 3(\bar{e}_B^i \cdot \bar{R}^i)\bar{R}^i\right] \tag{3.7.1}$$

where $\bar{V}_M^b$ is the vehicle magnetic moment and $\bar{e}_B$ is the geomagnetic dipole axis. The latter, for simplicity, is assumed to be co-axial with the earth north pole.

Substituting the expression of $\overset{*}{C}_{ib}$ from (3.3.30) into $\bar{L}_M^b$, we find $\overset{*}{Q}\bar{L}_M^b$ can be written in the form of

$$\overset{*}{Q}\bar{L}_M^b = \overset{*}{Q}_p(\tau_0)\ \bar{G}_2(\tau_1) \tag{3.7.2}$$

where

$$\overset{*}{Q}_p(\tau_0) = \overset{*}{Q}(\bar{V}_M^b \times)\ \overset{*}{C}_{ib}$$

$$= \overset{*}{Q}(\bar{V}_M^b \times)\left[\overset{*}{C}{}_1' \cos(2Rt) + \overset{*}{C}{}_2' \sin(2Rt) + \overset{*}{C}{}_3'\right]$$

$$\bar{G}_2(\tau_1) = \frac{\mu_B}{R^5}\left[R^2 \bar{e}_B^i - 3(\bar{e}_B^i \cdot \bar{R}^i)\bar{R}^i\right] \tag{3.7.3}$$

By expanding $\overset{*}{C}_1'$, $\overset{*}{C}_2'$, $\overset{*}{C}_3'$ into Fourier series, we see that $\overset{*}{\alpha} \bar{L}_M^b$ can be analytically integrated in terms of τ_0. Eq. (3.7.2) corresponds to (3.5.26) in section 3.5 and the asymptotic formulations can be easily applied.

Numerical Example

A rigid body satellite perturbed by the geomagnetic torque is simulated with the same satellite which flies in in the same orbit as given in section 3.6. In addition, we suppose that the vehicle carries a magnetic dipole $\bar{V}_M$, which is aligned with the body x-axis,

$$\bar{V}_M^{\,b} = (3, 0, 0) \quad ft \cdot amp \cdot sec$$

At mean time, the value of the geomagnetic field is assumed to be :

$$\mu_B = 22.2 \times 10^{24} \quad \frac{slug - ft^4}{sec^3 - amp}$$

Using the above numbers, we simulated the satellite dynamics by direct integration and by the asymptotic approach. Table 3.7.1 lists all the runs tried. The errors of each case are plotted in Fig. 3.7.1 through Fig. 3.7.12. Similar conclusions as given in the gravity gradient case can also be reached for the case of geomagnetic torque.

CHAPTER 4

PREDICTION OF ATTITUDE MOTION FOR A CLASS OF DUAL SPIN SATELLITE

4.1 Introduction

In chapter 3, a technique for speeding up the prediction of a rigid body satellite attitude motion was developed. However, the limitation that requires the satellite to be a rigid body seems severe, because many satellites in operation today have one or several high speed fly-wheels mounted onboard for the control or stabilization of their attitude motion. The combination of the vehicle and its flywheels sometimes is referred to as a dual spin satellite. Therefore, it seems desirable to expand our prediction method for handling the dual spin case as well. In doing so, it turns out that it is not difficult to modify our formulations to include the dual spin satellite, if the following conditions hold:

1. The angular velocity $\tilde{\omega}$ is a periodic function when there is no external torque.
2. A torque-free analytic solution of the system is possible.
3. External torques are small.

However, the dynamic characteristics of a dual spin

satellite in general are not clear yet as far as conditions one and two are concerned. Although we believe condition two might be relaxed by some further research, they are beyond the scope of this effort.

As a demonstration example for handling a dual spin case, we consider a special class of dual spin satellites: that is, a vehicle having a single fly-wheel which is mounted along one of the vehicle body principal axes. The satellite is allowed to have an arbitrary initial condition and to move in an arbitrary elliptic orbit.

In what follows, the rotational dynamics of a dual spin body are first discussed. An excellent reference on this subject is by Leimanis [22]. Later, the torque free solution - which serves as a nominal trajectory - for a class of dual spin satellites is presented. This torque-free solution was first given by Leipholz [23] in his study of the attitude motion of an airplane with a single rotary engine. Then, in section 4.4, an asymptotic formulation for a dual spin satellite is discussed and two sets of numerical simulations are presented.

4.2 Rotational Dynamics of a Dual Spin Satellite

For a dual spin satellite, we assume that the relative motion between fly-wheels and the vehicle do not alter the overall mass distribution of the combination. Thus, for convenience, the total angular momentum of the system about its center of mass can be resolved into two components. They are: $\bar{H}$, the angular momentum due to the rotational motion of the whole system regarded as a rigid body, and $\bar{H}_\omega$ the angular momentum of the fly-wheels with respect to the satellite.

The rotational motion of a dual spin satellite is, therefore, described by:

$$\frac{d^i}{dt}(\bar{H} + \bar{H}_\omega) = \bar{M}^i$$

$$\text{or} \quad \frac{d}{dt}(\overset{*}{I}_m \bar{\omega} + \bar{H}_\omega) = \bar{M}^i \tag{4.2.1}$$

where $\overset{*}{I}_m$ is the moment of inertia of the combination (vehicle and wheels), and $\bar{M}^i$ is the external disturbing torque.

Applying Coriolis law, we can transfer the above equation into body fixed coordinates 'b',

$$\overset{*}{I}_m \frac{d\bar{\omega}^b}{dt} + \bar{\omega}^b \times \overset{*}{I}_m \bar{\omega}^b + \frac{d\bar{H}_\omega^b}{dt} + \bar{\omega}^b \times \bar{H}_\omega^b = \bar{M}^b \tag{4.2.2}$$

Further, by assuming that the wheels have constant angular velocities with respect to the vehicle,

$$\frac{d\bar{H}_\omega^b}{dt} = 0$$

Hence

$$\overset{*}{I}_m \frac{d\bar{\omega}_b}{dt} + (\bar{\omega}^b \times)(\overset{*}{I}_m \bar{\omega}^b + \bar{H}_\omega^b) = \bar{M}^b \tag{4.2.3}$$

This equation is equivalent to Euler's equation in a rigid body case.

For satellite orientations, because Euler symmetric parameters and the angular velocity $\bar{\omega}$ are related kinematically, the equation remains the same for a dual spin satellite, that is,

$$\frac{d\bar{\beta}}{dt} = \frac{1}{2}[\overset{*}{\omega}]\bar{\beta} \tag{4.2.4}$$

4.3 The Torque-free Solution

The motion of a dual spin satellite with a single fly-wheel mounted along one of the body principal axes can be analytically solved if there is no external torque acting on the vehicle.

(A) Solution to Euler's equations

Without loss of generality, suppose the flywheel is aligned with the principal x-axis. By (4.2.3), the Euler's equations are

$$I_x \dot{\omega}_x = (I_y - I_z)\,\omega_y\,\omega_z \tag{4.3.1}$$

$$I_y \dot{\omega}_y = (I_z - I_x)\,\omega_x\,\omega_z - \omega_z\,h \tag{4.3.2}$$

$$I_z \dot{\omega}_z = (I_x - I_y)\,\omega_x\,\omega_y + \omega_y\,h \tag{4.3.3}$$

where 'h' is the angular momentum of the wheel with respect to the vehicle.

If eq. (4.3.2) is divided by (4.3.1), we have

$$I_y (I_y - I_z)\,\omega_y\,d\omega_y = [\,(I_z - I_x) I_x\,\omega_x - h I_x]\,d\omega_x$$

Integrating this equation and solving for ω_y in terms of ; the result is :

$$\omega_y = \pm \sqrt{\frac{(I_z - I_x) I_x\,\omega_x^2 - 2 h I_x\,\omega_x}{I_y (I_y - I_z)} + C_y} \tag{4.3.4}$$

where C_y is a constant which is yet to be determined.

Similarly, using eq.(4.3.3) and (4.3.1), ω_z can be also expressed in terms of ω_x ; the result is ;

$$\omega_z = \pm\sqrt{\frac{(I_x - I_y) I_x \omega_x^2 + 2 I_x h \omega_x}{I_z (I_y - I_z)} + C_z} \quad (4.3.5)$$

where C_z is another constant.

In case that there is no external torque, we know that the rotational kinetic energy T and the total angular momentum H_T of the system must remain constant. i.e.,

$$I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 + h \omega_{wheel} = 2T \quad (4.3.6)$$

and

$$(I_x \omega_x + h)^2 + I_y^2 \omega_y^2 + I_z^2 \omega_z^2 = H_T^2 \quad (4.3.7)$$

Equations (4.3.6) and (4.3.7) can be derived by integrating $[\omega_x \cdot (4.3.1) + \omega_y \cdot (4.3.2) + \omega_z (4.3.3)]$

and $[I_x \omega_x \ (4.3.1) + I_y \omega_y \ (4.3.2) + I_z \omega_z \ (4.3.3)]$, respectively.

Having determined the angular momentum H_T and kinetic energy T, one can calculate the constants c_y and c_z by substituting (4.3.4) and (4.3.5) into (4.3.6) and (4.3.7); they are ;

$$c_y = H_T^2 - h^2 - I_z\,(2T - h\,\omega_{wheel})$$
$$c_z = H_T^2 - h^2 - I_y\,(2T - h\,\omega_{wheel}) \qquad (4.3.8)$$

or ω_y and ω_z can be rewritten as ;

$$\omega_y = \pm\sqrt{\frac{I_x}{I_y - I_z}\,P_y}$$
$$\omega_z = \pm\sqrt{\frac{I_x}{I_z - I_y}\,P_z} \qquad (4.3.9)$$

where

$$P_y = \frac{I_z - I_x}{I_y}\,\omega_x^2 - \frac{2h}{I_y}\,\omega_x + c_y\,\frac{I_y - I_z}{I_x}$$

$$P_z = \frac{I_y - I_x}{I_z}\,\omega_x^2 - \frac{2h}{I_z}\,\omega_x + c_z\,\frac{I_z - I_y}{I_x}$$

(4.3.10)

To find ω_x, one can use (4.3.9), the solutions ω_y and ω_z, with (4.3.1) to eliminate ω_y and

ω_z , such that an equation for can be obtained.

$$\frac{d\omega_x}{dt} = \pm\sqrt{-P_y P_z} \tag{4.3.11}$$

or

$$t = \pm \int_{\omega_x(0)}^{\omega_x} \frac{d\omega_x}{\sqrt{-P_y P_z}} \tag{4.3.12}$$

Suppose that $-P_y P_z$ has roots of ω_1 , ω_2 , ω_3 and ω_4 in a descending order, then

$$t = \pm \int_{\omega_x(0)}^{\omega_x} \frac{d\omega_x}{\sqrt{-\left(\frac{I_y - I_x}{I_z}\right)\left(\frac{I_z - I_x}{I_y}\right)(\omega_x-\omega_1)(\omega_x-\omega_2)(\omega_x-\omega_3)(\omega_x-\omega_4)}} \tag{4.3.13}$$

This integral can be easily transformed into an elliptical integral; i.e. eq. (4.3.13) is equivalent to

$$t = m\int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{1-k^2 \sin^2\phi}} \tag{4.3.14}$$

where

$$m = \frac{2\sqrt{I_y \cdot I_z}}{\sqrt{(I_y - I_x)(I_z - I_x)(\omega_3-\omega_1)(\omega_4-\omega_2)}} \tag{4.3.15}$$

$$k^2 = \frac{\omega_4 - \omega_3}{\omega_4 - \omega_2} \cdot \frac{\omega_2 - \omega_1}{\omega_3 - \omega_1} \tag{4.3.16}$$

and if $\omega_4 \le \omega_x(0) \le \omega_3$, then

$$\omega_x = \frac{\omega_4(\omega_3-\omega_1)+\omega_1(\omega_4-\omega_3)\sin^2\phi}{\omega_3-\omega_1+(\omega_4-\omega_3)\sin^2\phi} \qquad (4.3.17)$$

and if $\omega_2 \le \omega_x(0) \le \omega_1$, then

$$\omega_x = \frac{\omega_2(\omega_3-\omega_1)-\omega_3(\omega_2-\omega_1)\sin^2\phi}{\omega_3-\omega_1-(\omega_2-\omega_1)\sin^2\phi} \qquad (4.3.18)$$

(B) Euler Symmetric Parameters

Since the equations of Euler symmetric parameters remain unchanged whether a space vehicle has fly-wheels or not,

$$\dot{\bar{\beta}} = \frac{1}{2}[\overset{*}{\omega}]\bar{\beta} \qquad (4.3.19)$$

Once the angular velocity $\bar{\omega}$ is obtained, this equation for the Euler symmetric parameters can be solved by a similar procedure as discussed in section 3.3. The result can be summarized in follows.

$$\begin{pmatrix}\beta_0(t)\\ \beta_1(t)\\ \beta_2(t)\\ \beta_3(t)\end{pmatrix} = \begin{bmatrix} E_1\cos(P_1+Rt) & 0 & -E_1\sin(P_1+Rt) & 0 \\ 0 & E_2\cos(P_2+Rt) & 0 & E_2\sin(P_2+Rt) \\ E_1\sin(P_1+Rt) & 0 & E_1\cos(P_1+Rt) & 0 \\ 0 & -E_2\sin(P_2+Rt) & 0 & E_2\cos(P_2+Rt) \end{bmatrix} \begin{pmatrix}\beta_0(0)\\ \beta_1(0)\\ \beta_2(0)\\ \beta_3(0)\end{pmatrix}$$

(4.3.20)

C-2

where

$$E_1 = \left[\frac{H_T + I_y\,\omega_y(t)}{H_T + I_y\,\omega_y(0)} \right]^{\frac{1}{2}}$$

$$E_2 = \left[\frac{H_T - I_y\,\omega_y(t)}{H_T - I_y\,\omega_y(0)} \right]^{\frac{1}{2}}$$

$$P_1 = 0.5\ \pi(t) - R\,t$$

$$P_2 = 0.5\ \Psi(t) - R\,t$$

$$R = \frac{\pi(T\omega)}{2\,T\omega}$$

and

$$\pi(t) = \int_0^t \frac{(2T - H_T\,\omega_y + h\,\omega_x}{H_T + I_y\,\omega_y}\,dt$$

$$\Psi(t) = \int_0^t \frac{(2T - H_T\,\omega_y + h\,\omega_x)}{H_T - I_y\,\omega_y}\,dt$$

$\Pi(t)$ and $\Psi(t)$ are two elliptic integrals, T and H_T are the kinetic energy and angular momentum of the system, and they are given by equations (4.3.6) and (4.3.7). T_ω is the period of the angular velocity $\bar{\omega}$, and h is the angular momentum of the fly-wheel with respect to the vehicle.

4.4. Asymptotic Solution and Numerical Results

Once the torque-free nominal solution for a dual spin satellite is obtained, the basic procedures of the asymptotic approach, described in section 3.5 for a rigid body satellite, can also be applied to a dual spin case. In order to include the gyro effect due to the fly-wheel, a few equations in section 3.5 have to be changed.

Equation (3.5.1) has to be replaced by

$$\overset{*}{I}_m \dot{\bar{\omega}} + [\bar{\omega}\times][\overset{*}{I}_m \bar{\omega} + \bar{H}_\omega] = \varepsilon^2 T_1 + \varepsilon^3 T_2 + \cdots \tag{4.4.1}$$

where $\bar{H}_\omega$ is the angular momentum of the flywheels with respect to the vehicle.

Equation (3.5.2) is replaced by

$$\overset{*}{I}_m \dot{\bar{\omega}}_N + [\bar{\omega}_N \times][\overset{*}{I}_m \bar{\omega}_N + \bar{H}_\omega] = 0 \tag{4.4.2}$$

and equation (3.5.5) has to be changed.

$$\overset{*}{A}(t) = -\overset{*}{I}_m \left[(\bar{\omega}_N \times) \overset{*}{I}_m - (\overset{*}{I}_m \bar{\omega}_N + \bar{H}_\omega)\times \right] \quad (4.4.3)$$

Of course, $\bar{\omega}_N(t)$ and the transition matrix $\Phi_\beta(t, o)$ arise from (4.3.9), (4.3.17) and (4.3.20). They are the reference trajectory for a dual spin case.

Numerical Simulations

For demonstrating the correctness and accuracy of predicting the attitude motion in the case of a dual spin satellite, we select a numerical example with the following data;

Satellite moment of inertia

I_x = 30 slug-ft^2

I_y = 25 slug-ft^2

I_z = 16 slug-ft^2

A fly-wheel is mounted along the body fixed x-axis of the vehicle with an angular momentum h with respect to the vehicle,

h = 0.2 slug-ft^2/sec

We assume that the satellite is in an elliptic ic orbit with

eccentricity e = 0.16

inclination i = 0

orbital period = 10,000 sec

The initial conditions are:

Angular velocity $\bar{\omega}(0)$

$\omega_x = 0.03$ rad/sec

$\omega_y = 0.01$ rad/sec

$\omega_z = 0.001$ rad/sec

Euler symmetric parameters $\bar{\beta}(0)$

$\beta_0 = 0.7071$

$\beta_1 = 0.1031$

$\beta_2 = 0.1065$

$\beta_3 = 0.6913$

For these numbers, the small parameter ε of the problem, defined as the ratio of orbital and attitude frequencies, is about

$$\varepsilon = \frac{\text{orbital frequency}}{\text{attitude frequency}} = \frac{2\pi/10{,}000}{2\pi/269} \approx 0.027$$

This dual spin satellite is first assumed to be disturbed by the gravity gradient torque only. The dynamics are simulated both by direct integration and the asymptotic approach. The results are summarized in Table 4.4.1. Also the simulation errors are presented in Fig. 4.4.1 through 4.4.10; they are given in the same way as in section 3.5 for a rigid body satellite.

Next, the dual spin satellite is assumed to be disturbed by the geomagnetic torque only. A magnetic dipole is placed onboard, with a strength of $\vec{V}_M^b =$ (3,0,0) ft - amp - sec, which interacts with the earth's magnetic field of strength

$$\mu_B = 22.2 \times 10^{24} \frac{slug - ft^4}{sec^3 - amp}$$

Similarly, the attitude dynamics are simulated by direct numerical integration and the asymptotic approach The results are summarized in Table 4.4.2 and Fig. 4.4.11 through Fig. 4.4.20.

Conclusion

These two sets of data, one for gravity gradient torque the other for geomagnetic torque, show that our asymptotic approach is equally useful for a dual spin satellite as for a rigid body case, if the conditions listed in section 4.1 can be satisfied. The numerical advantage of saving computer time and the approximation error introduced by our asymptotic approach are of similar character as discussed for a rigid body case. The details are not repeated again.

CHAPTER 5

DESIGN OF A MAGNETIC ATTITUDE CONTROL SYSTEM USING MTS METHOD

5.1 Introduction

The interaction between the satellite body magnetic moment with the geomagnetic field produces a torque on the satellite. This torque, however, can be harnessed as a control force for the vehicle attitude motion. By installing one or several current-carrying coils onboard, it is possible to generate an adjustable magnetic moment inside the vehicle and thus a control torque for the satellite. This magnetic attitude control device, using only the vehicle-enviroment interaction, needs no fuel and has no moving parts, it may conceivably increase the reliabiltiy of a satellite. In recent years, it has received considerable attention.

To design such a system, nevertheless, is difficult, because the control torque is very small. Since the electric currents available to feed through the onboard coils are limited, the magnetic torque generated in this way is not large enough to correct satellite attitude motion in a short period of time. In fact, it is realized that one has to depend on the long term accumulating control effort of the geomagnetic interaction to bring

the vehicle into a desired orientation. For this reason, the system can not be studied readily by classic control design techniques.

However, this problem can be more efficiently analyzed in terms of the slow variational equation from the MTS approach. By casting the dynamics of the above system into an MTS formulation, the fast motion (attitude nutational oscillation) and the slow motion (amplitude variation of the nutation) can be separated. Even though the effect of the magnetic torque on the dynamics is very difficult to observe and comprehend in real time t, still, using the slow secular equation in terms of a slow clock, the control effect on the nutation amplitude change immediately becomes transparent.

In this chapter, we will analyze a magnetic attitude control system for a dual spin, earth-pointing satellite. For more imformation, the reader may refer to the works of Renard[39], Wheeler [40] and Alfriend [41].

5.2 Problem Formulation

The Problem

A dual spin satellite moves in a circular orbit; its antenna is required to point toward the center of the earth. A momentum wheel is assumed to be mounted

along the satellite pitch axis for control of the pitch motion.

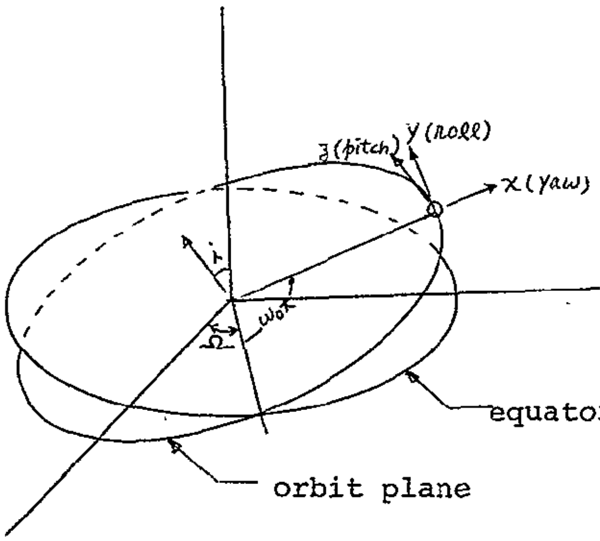


Fig. 5.1 Roll, yaw and pitch axes

For the above satellite a roll-yaw oscillation, called nutation, is possible. This is because its angular momentum vector may not be perfectly aligned with its angular velocity vector due to external disturbance or initial misalignent etc..

A magnetic control device, using the geomagnetic interaction, is to be designed to damp out the nutational oscillation as well as to keep vehicle's

angular momentum perpendicular to the orbit plane.

Equations of Motion

The Euler's equations for a dual spin satellite are given by (4.2.3), they are ;

$$
\begin{aligned}
I_x \dot{\omega}_x + (I_z - I_y)\,\omega_y\,\omega_z + h\,\omega_y &= M_x \\
I_y \dot{\omega}_y + (I_x - I_z)\,\omega_x\,\omega_z - h\,\omega_x &= M_y \\
I_z \dot{\omega}_z + (I_y - I_x)\,\omega_x\,\omega_y &= M_z
\end{aligned}
\qquad (5.2.1)
$$

where 'h' is the angular momentum of the fly-wheel and $\bar{M}$ is the external torque on the vehicle.

Assuming ψ , ϕ and θ are the Euler angles which correspond to yaw, roll and pitch for concatenated $x-y_1-z_2$ rotations, indicated by

$$
\underset{(x,y,z)}{\begin{pmatrix}\text{trajectory}\\ \text{frame}\end{pmatrix}} \xrightarrow[\text{by amount of } \psi]{\text{rotation about } x} (x_1,y_1,z_1) \xrightarrow[\text{by amount of } \phi]{\text{rotation about } y_1}
$$

$$
(x_2,y_2,z_2) \xrightarrow[\text{by amount of } \theta]{\text{rotation about } z_2} \overset{\begin{pmatrix}\text{body}\\ \text{frame}\end{pmatrix}}{(x_b,y_b,z_b)}
$$

Then, a kinematic relation between the angular velocity and the derivatives of the Euler angles can be written as [42]:

$$\bar{\omega} = \dot{\psi}\,\hat{e}_x + \dot{\phi}\,\hat{e}_{y_1} + \dot{\theta}\,\hat{e}_{z_2} + \omega_0 t\,\hat{e}_z \qquad (5.2.2)$$

where $\hat{e}_x$, $\hat{e}_z$ are unit vectors, $\hat{e}_x$ is in the direction of vehicle position vector and $\hat{e}_z$ is normal to the orbital plane. The term $(\omega_0 t)\,\hat{e}_z$ describes the rotation motion of the satellite in a circular orbit.

Transforming $\hat{e}_x$, $\hat{e}_z$ and $\hat{e}_{y_1}$ into the body fixed coordinates and using small angles assumption, equation (5.2.2) can be written as:

$$\bar{\omega}^b = \begin{bmatrix} \dot{\psi}\cos\phi\cos\theta + \dot{\phi}\sin\theta - \omega_0\cos\psi\sin\phi\cos\theta + \omega_0\sin\psi\sin\theta \\ -\dot{\psi}\cos\phi\sin\theta + \dot{\phi}\cos\theta + \omega_0\cos\psi\sin\phi\sin\theta + \omega_0\sin\psi\cos\theta \\ \dot{\psi}\sin\phi + \dot{\theta} + \omega_0\cos\psi\cos\phi \end{bmatrix} \qquad (5.2.3)$$

$$= \begin{bmatrix} \dot{\psi} - \omega_0\phi \\ \dot{\phi} + \omega_0\psi \\ \dot{\theta} + \omega_0 \end{bmatrix} \qquad (5.2.4)$$

Substitute (5.2.4) into Euler's equation (5.2.1), we have :

$$I_x(\ddot{\psi} - \omega_0\dot{\varphi}) + (I_z - I_y)(\dot{\varphi} + \omega_0\psi)(\omega_0 + \frac{h}{I_z - I_y}) = M_x$$
$$I_y(\ddot{\varphi} + \omega_0\dot{\psi}) + (I_x - I_z)(\dot{\psi} - \omega_0\varphi)(\omega_0 - \frac{h}{I_x - I_z}) = M_y$$
$$I_z\ddot{\theta} = M_z \qquad (5.2.5)$$

Earth's Magnetic Field

The geomagnetic field $\bar{B}$ can be approximately represented by a magnetic dipole as discussed in section 3.4.

$$\bar{B} = \frac{\mu_B}{R^5}\left[R^2\bar{e}_B - 3(\bar{e}_B \cdot \bar{R})\bar{R}\right] \qquad (5.2.6)$$

where μ_B is a constant of the geomagnetic field ($\mu_B = 8.1 \times 10^{25}$ gauss-cm^3), $\bar{e}_B$ is the unit vector along the dipole axis and $\bar{R}$ is the vehicle position vector. Using Fig. 5.1, $\bar{e}_B$ can be expressed in the axes of $\hat{e}_x, \hat{e}_y, \hat{e}_z$ (called trajectory axes):

$$\bar{e}_B = \sin\omega_0 t \sin i \;\hat{e}_x + \cos\omega_0 t \sin i \;\hat{e}_y + \cos i \;\hat{e}_z \qquad (5.2.7)$$

where 'i' is the inclination of the orbit. Substitute (5.2.7) into (5.2.6), we have:

$$\bar{B}^{traj} = \frac{\mu_B}{R^5}\left[-2\sin\omega_0 t \sin i \;\hat{e}_x + \cos\omega_0 t \sin i \;\hat{e}_y + \cos i \;\hat{e}_z\right] \qquad (5.2.8)$$

Further, because the satellite is in a circular orbit

$$\frac{\mu}{R^3} = \omega_0^2$$

and if the vehicle is well controlled, the body axes of the satellite should be closely aligned with the trajactory axes; therefore,

$$\bar{B}^b \approx \bar{B}^{traj} = \frac{\omega_0^2 \mu_B}{\mu} \begin{bmatrix} -2 \sin\omega_0 t \sin i \\ \cos\omega_0 t \sin i \\ \cos i \end{bmatrix} \tag{5.2.9}$$

or

$$\bar{B}^b = \begin{bmatrix} -2 B_0 \sin\omega_0 t \\ B_0 \cos\omega_0 t \\ B_1 \end{bmatrix} \tag{5.2.9'}$$

where

$$B_0 = \frac{\omega_0^2 \mu_B}{\mu} \sin i$$

$$B_1 = \frac{\omega_0^2 \mu_B}{\mu} \cos i$$

The Control Torque

Assuming $\bar{V}_M$ to be the magnetic moment of the vehicle, the torque experienced by the satellite due to the geomagnetic field is

$$\bar{M} = \bar{V}_M \times \bar{B} \tag{5.2.10}$$

For the control of the vehicle's attitude, let us pre-specify the control torque $\bar{M}$ as

$$\bar{M}_{desired} = -k_2 \begin{bmatrix} I_x \dot{\psi} \\ I_y \dot{\phi} \\ 0 \end{bmatrix} - k_1 \begin{bmatrix} h\phi \\ -h\psi \\ 0 \end{bmatrix} \tag{5.2.11}$$

$\bar{M}_{desired}$ represents a desirable control law, which may not be possible to implement. $\bar{M}_{desired}$ is two-dimensional, because the pitch motion is controlled by the momentum wheel, which is excluded in this analysis. The first term in the above equation, $-k_2(I_x\dot{\psi}, I_y\dot{\phi}, 0)$ reflects elimination of excessive angular momentum due to the perturbed vehicle body rotational motion and thus damps the nutational motion. The second term is to control the vehicle orientation through the gyro effect of the momentum wheel.

Let us take the cross product of equation (5.2.10) by $\bar{B}$,

$$\bar{B} \times \bar{M} = \bar{B} \times (\bar{V}_M \times \bar{B})$$

$$= |\bar{B}|^2 \bar{V}_M - \bar{B} (\bar{V}_M \cdot \bar{B}) \qquad (5.2.12)$$

The above equation may be satisfied if we pick

$$\bar{V}_M = \frac{\bar{B} \times \bar{M}}{|B_E|^2} \qquad (5.2.13)$$

In this case $\bar{V}_M$ will be perpendicular to $\bar{B}$ and also to $\bar{M}$. It can be proved that this $\bar{V}_M$ is the smallest magnetic moment required for generating a known torque $\bar{M}$. Substitute (5.2.11) into (5.2.13) to give:

$$\bar{V}_M = \frac{1}{|\bar{B}|^2} \begin{bmatrix} -B_z (-k_2 I_y \dot{\phi} + k_1 h \psi) \\ B_z (-k_2 I_x \dot{\psi} - k_1 h \phi) \\ B_x (-k_2 I_y \dot{\phi} + k_1 h \psi) \\ \quad - B_y (-k_2 I_x \dot{\psi} - k_1 h \phi) \end{bmatrix} \qquad (5.2.14)$$

We see that $\bar{V}_M$ is three dimensional. Thus in order to implement the above control law, it requires three electric current-carrying coils and their supplemental equipment to be mounted orthogonally to each other in the vehicle.

However, let us suppose that there is a weight and space restriction. We have to limit ourselves to use one coil only. By putting a single coil along the pitch

axis, the first and second components of equation (5.2.14) are eliminated; the control law now is

$$\bar{V}_M^b = \frac{1}{|\bar{B}|^2}\begin{bmatrix} 0 \\ 0 \\ B_x(-k_2 I_y \dot{\phi} + k_1 h \psi) - B_y(-k_2 I_x \dot{\psi} - k_1 h \phi) \end{bmatrix} \tag{5.2.15}$$

The corresponding torque generated by this control law will be:

$$\begin{aligned} \bar{M}^b &= \bar{V}_M \times \bar{B} \\ &= \frac{[B_x(k_2 I_y \dot{\phi} - k_1 h \psi) - B_y(k_2 I_x \dot{\psi} + k_1 h \phi)]}{|\bar{B}|^2}\begin{bmatrix} B_y \\ -B_x \\ 0 \end{bmatrix} \end{aligned} \tag{5.2.16}$$

By substituting (5.2.16) and (5.2.9) into the equation of motion (5.2.5) we have:

$$\begin{aligned} & I_x \ddot{\psi} + \left(\frac{B_0^2 \cos^2 \omega_0 t \; k_2 I_x}{|\bar{B}|^2}\right)\dot{\psi} + \Bigg[\frac{B_0^2 \sin 2\omega_0 t \; k_2 I_y}{|\bar{B}|^2} - I_x \omega_0 \\ & + (I_3 - I_y)\omega_0 + h\Bigg]\dot{\phi} + \Bigg[\frac{-B_0^2 \sin 2\omega_0 t \; k_1 h}{|\bar{B}|^2} + \omega_0^2(I_3 - I_y) \\ & + h\omega_0\Bigg]\psi + \frac{B_0^2 \cos^2 \omega_0 t \; k_1 h}{|\bar{B}|^2}\phi = 0 \end{aligned} \tag{5.2.17a}$$

and

$$I_y \ddot{\phi} + \left[\frac{4 B_0^2 k_2 I_y \sin^2 \omega_0 t}{|\bar{B}|^2} \right] \dot{\phi}$$

$$+ \left[\frac{B_0^2 \sin 2\omega_0 t \; k_2 I_x}{|\bar{B}|^2} + I_y \omega_0 + \omega_0 (I_x - I_z) - h \right] \dot{\psi}$$

$$+ \left[-\omega_0^2 (I_x - I_z) + \omega_0 h + \frac{B_0^2 \sin 2\omega_0 t \; k_1 h}{|\bar{B}|^2} \right] \phi$$

$$- \frac{4 B_0^2 \sin^2 \omega_0 t}{|\bar{B}|^2} k_1 h \psi = 0 \qquad (5.2.17b)$$

These equations are linear, but can not be exactly solved, since the coefficients are time-varying. The quantities k_1 and k_2 should be picked such that the above system is stable.

5.3 System Analysis

Free Response Solution

The roll-yaw dynamics (5.2.5) are first determined without considering external torques. We see that the dynamics contain a fast nutational mode and a slow orbital mode.

The system (5.2.5), without the external torque $\bar{M}$,

has the characteristic equation:

$$\lambda^4 + \left[\omega_o^2 - \frac{(I_z - I_y)(I_x - I_z)\left(\omega_o + \frac{h}{I_z - I_y}\right)\left(\omega_o - \frac{h}{I_x - I_z}\right)}{I_x I_y} \right] \lambda^2$$

$$- \frac{\omega_o^2 (I_z - I_y)(I_x - I_z)\left(\omega_o + \frac{h}{I_z - I_y}\right)\left(\omega_o - \frac{h}{I_x - I_z}\right)}{I_x I_y} = 0 \tag{5.3.1}$$

The four characteristic roots are:

$\pm i \, \omega_o$ orbital mode

$\pm i \sqrt{\frac{[(I_z - I_y)\omega_o + h] \cdot [h - (I_x - I_z)\omega_o]}{I_x I_y}}$ nutational mode

Since ω_o (the orbital angular velocity) is much smaller than ω_{wheel} ($h = I_{wheel}\,\omega_{wheel}$), the angular velocity of the fly wheel, therefore the nutational mode can be simplified into $\pm \frac{i h}{\sqrt{I_x I_y}}$. We note that the frequency ratio of orbital mode and nutational mode is a very small number; that is $\omega_o \big/ \frac{h}{\sqrt{I_x I_y}} \ll 1$

Order of Magnitude Consideration

Let ε be a small parameter which is defined as the ratio of orbital frequency to the nutational frequency.

$$\varepsilon = \left| \frac{\omega_o}{\frac{h}{\sqrt{I_x I_y}}} \right| \ll 1 \tag{5.3.2}$$

Assume h, I_x, I_y and I_z are of order one, then the following terms are ordered as:

$$
\begin{aligned}
&h = O(1) \\
&\omega_0 = O(\varepsilon) \\
&B_0 = \omega_0^2 \mu_B \sin i / \mu = O(\varepsilon^2) \\
&B_1 = \omega_0^2 \mu_B \cos i / \mu = O(\varepsilon^2) \\
&|B| = O(\varepsilon^2)
\end{aligned}
\tag{5.3.3}
$$

The control gains k_1, k_2 are limited by the small current generating the torque and is therefore, necessarily small. We assume that k_1, $k_2 = O(\varepsilon)$.

For convenience, ε and its power can be used to indicate the order of each term. For instance ω_0 will be replaced by $\varepsilon\omega_0$. By doing so, equations (5.2.17) are :

$$
\begin{aligned}
&I_x \ddot{\psi} + \varepsilon \left[\frac{B_0^2 \cos^2 \varepsilon\omega_0 t}{|\bar{B}|^2} k_2 I_x \right] \dot{\psi} + h\dot{\phi} \\
&+ \varepsilon \left[\frac{B_0^2 \sin 2\varepsilon\omega_0 t \; k_2 I_y}{|\bar{B}|^2} - I_x \omega_0 + (I_z - I_y)\omega_0 \right] \dot{\phi} \\
&+ \varepsilon \left[\frac{-B_0^2 \sin 2\varepsilon\omega_0 t \; k_1 h}{|\bar{B}|^2} + \omega_0 h + \varepsilon \omega_0^2 (I_z - I_y) \right] \psi \\
&+ \varepsilon \left[\frac{B_0^2 \cos^2 \varepsilon\omega_0 t}{|\bar{B}|^2} k_1 h \right] \phi = 0
\end{aligned}
\tag{5.3.4}
$$

$$I_y\ddot{\phi} + \varepsilon\left[\frac{4B_0^2 k_2 I_y \sin^2\varepsilon\omega_c t}{|\bar{B}|^2}\right]\dot{\phi} - h\dot{\psi}$$

$$+ \varepsilon\left[\frac{B_0^2 \sin 2\varepsilon\omega_c t\, k_2 I_x}{|\bar{B}|^2} + I_y\omega_c + \omega_c(I_x - I_z)\right]\dot{\psi}$$

$$+ \left[\omega_c h - \varepsilon\,\omega_c^2(I_x - I_z) + \frac{B_0^2 \sin 2\omega_c\varepsilon t\, k_1 h}{|\bar{B}|^2}\right]\phi$$

$$+ \left[\frac{-4B_0^2 \sin^2\varepsilon\omega_c t}{|\bar{B}|^2} k_1 h\right]\psi = 0 \qquad (5.3.5)$$

From these equations, it is easy to see that all the control terms are at least one order less than the system dynamics; therefore the control does not influence the fast dynamics.

To simplify the notation, (5.3.4) and (5.3.5) can be re-written as:

$$I_x\ddot{\psi} + \varepsilon\, G_\psi\dot{\psi} + (h + \varepsilon D_\psi)\dot{\phi} + \varepsilon E_\psi \psi$$
$$+ \varepsilon F_\psi \phi = 0 \qquad (5.3.4')$$

$$I_y\ddot{\phi} + \varepsilon\, G_\phi\dot{\phi} + (-h + \varepsilon D_\phi)\dot{\psi} + \varepsilon E_\phi \phi$$
$$+ \varepsilon F_\phi \psi = 0 \qquad (5.3.5')$$

where G_ψ, G_ϕ, D_ψ ....etc. are defined by comparing (5.3.4′) to (5.3.4) and (5.3.5') to (5.3.5).

Multiple Time Scales Approach

For solving equation (5.3.4') and (5.3.5'), we use the MTS method as outlined in section 2.2, by expanding the dependent variables ψ and ϕ into asymptotic series in ε,

$$\psi = \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \cdots \qquad (5.3.6)$$

$$\varphi = \varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \cdots \qquad (5.3.7)$$

Further, the dimension of the independent variable t (time) is also expanded into multi-dimensions as given by (2.2.11), (2.2.12), and (2.2.13). Expressing equations (5.3.4') and (5.3.5') in new time dimensions, we have:

$$\begin{aligned} & I_x\left(\frac{\partial^2 \psi_0}{\partial \tau_0^2} + 2\varepsilon \frac{\partial^2 \psi_0}{\partial \tau_0 \partial \tau_1} + \varepsilon \frac{\partial^2 \psi_1}{\partial \tau_0^2} + \cdots\right) \\ & + \varepsilon\, d_\psi \left(\frac{\partial \psi_0}{\partial \tau_0} + \varepsilon \frac{\partial \psi_0}{\partial \tau_1} + \varepsilon \frac{\partial \psi_1}{\partial \tau_0} + \cdots\right) \\ & + (h + \varepsilon D_\psi)\left(\frac{\partial \varphi_0}{\partial \tau_0} + \varepsilon \frac{\partial \varphi_0}{\partial \tau_1} + \varepsilon \frac{\partial \varphi_1}{\partial \tau_0} + \cdots\right) \\ & + \varepsilon E_\psi (\psi_0 + \varepsilon \psi_1 + \cdots) - \varepsilon F_\psi (\varphi_0 + \varepsilon \phi_1 + \cdots) \\ & = 0 \end{aligned} \qquad (5.3.8)$$

and

$$I_y\left(\frac{\partial^2\varphi_0}{\partial\tau_0^2} + 2\varepsilon\frac{\partial^2\varphi_0}{\partial\tau_0\partial\tau_1} + \varepsilon\frac{\partial^2\varphi_1}{\partial\tau_0^2} + \cdots\right)$$

$$+ \varepsilon\, G_\varphi\left(\frac{\partial\varphi_0}{\partial\tau_0} + \varepsilon\frac{\partial\varphi_0}{\partial\tau_1} + \varepsilon\frac{\partial\varphi_1}{\partial\tau_0} + \cdots\right)$$

$$+ (-h + \varepsilon D_\varphi)\left(\frac{\partial\psi_0}{\partial\tau_0} + \varepsilon\frac{\partial\psi_0}{\partial\tau_1} + \varepsilon\frac{\partial\psi_1}{\partial\tau_0} + \cdots\right)$$

$$+ \varepsilon E_\varphi(\varphi_0 + \varepsilon\varphi_1 + \cdots) + \varepsilon F_\varphi(\psi_0 + \varepsilon\psi_1 + \cdots)$$

$$= 0 \qquad (5.3.9)$$

By equating the terms of like powers of ε in the above equations, we have:

Terms of ε^0

$$I_x\frac{\partial^2\psi_0}{\partial\tau_0^2} + h\frac{\partial\varphi_0}{\partial\tau_0} = 0$$

$$I_y\frac{\partial^2\varphi_0}{\partial\tau_0^2} - h\frac{\partial\psi_0}{\partial\tau_0} = 0 \qquad (5.3.10)$$

The solutions are ;

$$\psi_0 = p\,\exp\left[\frac{-hi\tau_0}{\sqrt{I_x I_y}}\right] + q\,\exp\left[\frac{hi\tau_0}{\sqrt{I_x I_y}}\right] + r$$

$$\varphi_0 = \frac{iI_x p}{\sqrt{I_x I_y}}\exp\left[\frac{-hi\tau_0}{\sqrt{I_x I_y}}\right] - \frac{iI_x q}{\sqrt{I_x I_y}}\exp\left[\frac{hi\tau_0}{\sqrt{I_x I_y}}\right] + s$$

(5.3.11)

Where p, q, r, and s are not functions τ_0 , but they may be functions of $\tau_1, \tau_2 \cdots$etc. They are yet to be determined.

Terms of ε^1:

$$I_x \frac{\partial^2 \psi_1}{\partial \tau_0^2} + h \frac{\partial \varphi_1}{\partial \tau_0} = -2 I_x \frac{\partial^2 \psi_0}{\partial \tau_0 \partial \tau_1} - G_\psi \frac{\partial \psi_0}{\partial \tau_0} - h \frac{\partial \varphi_0}{\partial \tau_1} - D_\psi \frac{\partial \varphi_0}{\partial \tau_0} - E_\psi \psi_0 - F_\psi \varphi_0$$

$$I_y \frac{\partial^2 \varphi_1}{\partial \tau_0^2} - h \frac{\partial \psi_1}{\partial \tau_0} = -2 I_y \frac{\partial^2 \varphi_0}{\partial \tau_0 \partial \tau_1} - G_\varphi \frac{\partial \varphi_0}{\partial \tau_0} + h \frac{\partial \psi_0}{\partial \tau_1} - D_\varphi \frac{\partial \psi_0}{\partial \tau_0} - E_\varphi \varphi_0 - F_\varphi \psi_0 \tag{5.3.12}$$

Substituting the solutions of ψ_0 and φ_0 from (5.3.11) into (5.3.12), we have:

$$I_x \frac{\partial^2 \psi_1}{\partial \tau_0^2} + h \frac{\partial \varphi_1}{\partial \tau_0} = \exp\left[-i N_u \tau_0\right] U_\psi + \exp\left[i N_u \tau_0\right] V_\psi + W_\psi$$

$$I_y \frac{\partial^2 \varphi_1}{\partial \tau_0^2} - h \frac{\partial \psi_1}{\partial \tau_0} = \exp\left[-i N_u \tau_0\right] U_\varphi + \exp\left[i N_u \tau_0\right] V_\varphi + W_\varphi \tag{5.3.13}$$

where

$$
\begin{aligned}
N_u &= \frac{h}{\sqrt{I_x I_y}} \\
\mathcal{U}_\psi &= N_u i \left[I_x \frac{\partial p}{\partial \tau_1} + \left(G_\psi - F_\psi \frac{I_x}{h} + \frac{D_\psi I_x i}{\sqrt{I_x I_y}} + \frac{E_\psi i}{N_u} \right) p \right] \\
\mathcal{V}_\psi &= -N_u i \left[I_x \frac{\partial q}{\partial \tau_1} + \left(G_\psi - F_\psi \frac{I_x}{h} - \frac{D_\psi I_x i}{\sqrt{I_x I_y}} - \frac{E_\psi i}{N_u} \right) q \right] \\
\mathcal{W}_\psi &= -h \frac{\partial s}{\partial \tau_1} - E_\psi r - F_\psi s \qquad\qquad (5.3.14) \\
\mathcal{U}_\phi &= -h \frac{\partial p}{\partial \tau_1} - \left(G_\phi \frac{h}{I_y} + F_\phi - D_\phi N_u i + \frac{E_\phi I_x i}{\sqrt{I_x I_y}} \right) p \\
\mathcal{V}_\phi &= -h \frac{\partial q}{\partial \tau_1} - \left(G_\phi \frac{h}{I_y} + F_\phi + D_\phi N_u i - \frac{E_\phi I_x i}{\sqrt{I_x I_y}} \right) q \\
\mathcal{W}_\phi &= h \frac{\partial r}{\partial \tau_1} - E_\phi s - F_\phi r
\end{aligned}
$$

Note $\mathcal{U}_\psi, \mathcal{V}_\psi$ ... etc. are all functions of τ_1, τ_2 only. And p, q are yet to be determined.

In order that $|\psi_1 / \psi_0|$ and $|\phi_1 / \phi_0|$ be bounded, the terms that produce secular effects in ψ_1, ϕ_1 must be eliminated. Equations (5.3.13) are linear; the transition matrix $\Phi(t, o)$ for $\frac{\partial \psi_1}{\partial \tau_0}$ and $\frac{\partial \phi_1}{\partial \tau_0}$ in (5.3.13) is

$$\Phi(t,0) = \begin{bmatrix} \cos N_u \tau_0 & -\sin N_u \tau_0 \\ \sin N_u \tau_0 & \cos N_u \tau_0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \exp(iN_u\tau_0) + \exp(-iN_u\tau_0) \; , & i\exp(iN_u\tau_0) - i\exp(-iN_u\tau_0) \\ -i\exp(iN_u\tau_0) + i\exp(-iN_u\tau_0) \; , & \exp(iN_u\tau_0) + \exp(-iN_u\tau_0) \end{bmatrix} \tag{5.3.15}$$

The solutions for $\frac{\partial \psi_1}{\partial \tau_0}$, $\frac{\partial \phi_1}{\partial \tau_0}$ with zero initial conditions are :

$$\begin{bmatrix} \frac{\partial \psi_1}{\partial \tau_0} \\ \frac{\partial \phi_1}{\partial \tau_0} \end{bmatrix} = \int_0^{\tau_0} \Phi(\tau_0,\sigma) \begin{bmatrix} \exp(-iN_u\sigma)\,U_\psi + \exp(iN_u\sigma)\,V_\psi + W_\psi \\ \exp(-iN_u\sigma)\,U_\phi + \exp(iN_u\sigma)\,V_\phi + W_\phi \end{bmatrix} \cdot d\sigma$$

$$= \frac{\Phi(\tau_0, 0)}{2}$$

$$\begin{bmatrix} (-U_\varphi + i\,U_\psi)\dfrac{\exp(-2iN_u\tau_0)}{2\,N_u} - (iV_\psi + V_\varphi)\dfrac{\exp(2iN_u\tau_0)}{2\,N_u} \\ + (iW_\psi - W_\varphi)\dfrac{\exp(-iN_u\tau_0)}{N_u} - (W_\varphi + i\,W_\psi)\dfrac{\exp(iN_u\tau_0)}{N_u} \\ + (V_\psi + U_\psi + V_\varphi i - U_\varphi i)\,\tau_0 \\ \\ (U_\psi + i\,U_\varphi)\dfrac{\exp(-2iN_u\tau_0)}{2\,N_u} + (V_\psi - i\,V_\varphi)\dfrac{\exp(2iN_u\tau_0)}{2\,N_u} \\ + (W_\psi + i\,W_\varphi)\dfrac{\exp(-iN_u\tau_0)}{N_u} + (W_\psi - i\,W_\varphi)\dfrac{\exp(iN_u t)}{N_u} \\ + (V_\varphi + U_\varphi - i\,V_\psi + i\,U_\psi)\,\tau_0 \end{bmatrix} \qquad (5.3.16)$$

Notice, the terms $(V_\psi + U_\psi + V_\varphi i - U_\varphi i)\,\tau_0$ and $(V_\varphi + U_\varphi - iV_\psi + iU_\psi)\,\tau_0$ will increase linearly with time. In order to have ψ_1 , φ_1 bounded, these two secular terms have to be set to zero. They are:

$$V_\psi + U_\psi + V_\varphi i - U_\varphi i = 0$$

$$V_\varphi + U_\varphi - i\,V_\psi + i\,U_\psi = 0 \qquad (5.3.17)$$

which are equivalent to

$$\mathcal{U}_\varphi + i\,\mathcal{U}_\psi = 0$$

$$V_\varphi - i\,V_\psi = 0 \tag{5.3.18}$$

Also, using the freedom of choosing r and s , we can set $W_\psi = W_\varphi = 0$, or

$$W_\psi = -h\frac{\partial s}{\partial \tau_1} - E_\psi r - F_\psi s = 0$$

$$W_\varphi = h\frac{\partial r}{\partial \tau_1} - E_\varphi s - F_\varphi r = 0 \tag{5.3.19}$$

Substitute for $\mathcal{U}_\varphi$, V_φ , W_ψ ...etc. from (5.3.14); equations (5.3.18) are, then:

$$(h + N_u I_x)\frac{\partial p}{\partial \tau_1} + \Big[\Big(G_\varphi \frac{h}{I_y} + F_\varphi - D_\varphi N_u i + \frac{E_\varphi I_x i}{\sqrt{I_x I_y}}\Big) + N_u\Big(G_\psi - F_\psi \frac{I_x}{h} + \frac{D_\psi I_x i}{\sqrt{I_x I_y}} + \frac{E_\psi i}{N_u}\Big)\Big] p = 0 \tag{5.3.20}$$

$$(h + N_u I_x)\frac{\partial q}{\partial \tau_1} + \Big[\Big(G_\varphi \frac{h}{I_y} + F_\varphi + D_\varphi N_u i - \frac{E_\varphi I_x i}{\sqrt{I_x I_y}}\Big) + N_u\Big(G_\psi - F_\psi \frac{I_x}{h} - \frac{D_\psi I_x i}{\sqrt{I_x I_y}} - \frac{E_\psi i}{N_u}\Big)\Big] q = 0 \tag{5.3.21}$$

where 'p' and 'q' are complex conjugate.

A first order MTS asymptotic solution for (5.3.4) and (5.3.5) thus has been constructed by using (5.3.11);

that is

$$\psi = p \exp(-N_u i \tau_0) + q \exp(N_u i \tau_0) + r$$

$$\phi = \frac{i I_x p}{\sqrt{I_x I_y}} \exp(-N_u i \tau_0) - \frac{i I_x q}{\sqrt{I_x I_y}} \exp(N_u i \tau_0) + s \tag{5.3.22}$$

where p, q are given by (5.3.20), and r, s are given by (5.3.19). All these equations are in terms of τ_1 - the slow time scale. In order to make ψ and ϕ decay in time, the control torque has to be selected such that it forces p,q,r,s to zero through the slow equations. The general problem in chapter 4 (with magnetic torque) is simpler in the present case because the perturbed motion is kept small by the control. Linearization therefore, seems adequate, leading to analytical solution.

Selection of Feedback control gains

We wish to select feedback gains k_1 and k_2 in system (5.3.4) and (5.3.5), such that the system is stable. However, because the control torque is small and the system is time-varying, the effect of the control upon the dynamics is not clear.

The above problem was then cast into an asymptotic formulation. By the MTS method, an approximate solution has been achieved as given by (5.3.22), through which the design problem can be interpreted in a more convenient way. That is because the approximate solution (5.3.22) describes a constant frequency oscillatory motion with

slowly time-varying amplitude p, q and biases r, s. p, q, r, and s are functions of the control torque and are given by (5.3.19) and (5.3.20). The problem can be considered differently in that the control torque is used to force the amplitude p, q and biases r, s to zero. In this way, since the fast nutational oscillation has been separated out, the selection of the feedback gains k_1 and k_2 in the slow equations (5.3.19) and (5.3.20) becomes much easier.

The solution for $p(\tau_1)$ [or $q(\tau_1)$] from (5.3.20) is

$$p = p_0 \exp\left\{ \frac{\dfrac{-C_\phi}{I_y} - \dfrac{F_\phi}{h} - \dfrac{C_\psi}{\sqrt{I_x I_y}} + \dfrac{F_\psi I_x}{h\sqrt{I_x I_y}}}{1 + \dfrac{I_x}{\sqrt{I_x I_y}}} \tau_1 + \text{Imaginary part} \right\} \tag{5.3.23}$$

Thus p and q will approach zero if

$$\frac{-C_\phi}{I_y} - \frac{F_\phi}{h} - \frac{C_\psi}{\sqrt{I_x I_y}} + \frac{F_\psi I_x}{h\sqrt{I_x I_y}} < 0$$

where C_ϕ, F_ϕ etc. are given by (5.3.4') and (5.3.5'). By substitution, (5.3.22) becomes

$$4 \sin^2 \omega_0 t \; (k_1 - k_2) + \frac{I_x \cos^2 \omega_0 t}{\sqrt{I_x I_y}} (k_1 - k_2) < 0$$

or equivalently, for p and q to be decaying, we require

$$k_1 < k_2 \qquad (5.3.24)$$

Also r and s are governed by (5.3.19), which is a second order linear differential equation with periodic coefficients. The necessary and sufficient condition for such a system to approach zero as τ_1 approaches infinity is that the eigenvalues of $\Phi(\tau_1+T, \tau_1)$ lie in a unit disc $|\lambda|<1$ where $\Phi(t,t_0)$ is the transition matrix of the system and 'T' is the period of the periodic coefficients. Equations (5.3.19) can be written as:

$$\begin{bmatrix} \dfrac{\partial s}{\partial \tau_1} \\ \\ \dfrac{\partial r}{\partial \tau_1} \end{bmatrix} = \begin{bmatrix} \dfrac{-F_\psi}{h}, & \dfrac{-E_\psi}{h} \\ \\ \dfrac{E_\phi}{h}, & \dfrac{F_\phi}{h} \end{bmatrix} \cdot \begin{bmatrix} s \\ \\ r \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-B_0^2 \cos^2 \omega_0 \tau_1 \cdot k_1}{|\vec{B}|^2}, & \dfrac{B_0^2 \sin 2\omega_0 \tau_1 \, k_1}{|\vec{B}|^2} - \omega_0 \\ \\ \omega_0 + \dfrac{B_0^2 \sin 2\omega_0 \tau_1 \, k_1}{|\vec{B}|^2}, & \dfrac{-4 B_0^2 \sin^2 \omega_0 \tau_1 \, k_1}{|\vec{B}|^2} \end{bmatrix} \cdot \begin{pmatrix} s \\ \\ r \end{pmatrix}$$

$$= \overset{*}{A}(\tau_1) \cdot \begin{pmatrix} s \\ r \end{pmatrix} \qquad (5.3.25)$$

Note that $\Phi(t, t_c)$ does not depend on k_2 .

5.4 An Example

With the problem described in section 5.2, suppose we have a satellite with parameters as follows:

Moment of inertia

$$I_x = 120 \quad \text{slug - ft}^2$$

$$I_y = 100 \quad \text{slug - ft}^2$$

$$I_z = 150 \quad \text{slug - ft}^2$$

Angular momentum of the fly-wheel (along pitch axis)

$$h = 4 \quad \text{slug - ft}^2 / \text{sec}$$

and satellite is in an orbit of:

eccentricity $e = 0$

inclination $i = 20^\circ$

period $= 10{,}000$ sec

The small parameter ε , by (5.3.3), is then

$$\varepsilon = \frac{\omega_{orbit}}{\frac{h}{\sqrt{I_x I_y}}} \approx 0.017$$

The value of ε can give a rough indication as to the degree of accuracy of the approximate solution.

Since the roll-yaw motion can be approximated by (5.3.22)

$$\psi = p \exp(-Nu\, i\, t) + q \exp(Nu\, i\, t) + r$$

$$\phi = \frac{i\, Nu\, I_x\, p}{h} \exp(-Nu\, i\, t) - i \frac{Nu\, I_x\, q}{h} \exp(Nu\, i\, t) + s$$

where p, q are given by (5.3.21) and r, s are given by (5.3.25). For p, q to decay, we require:

$$k_1 < k_2$$

and for r, s to be decaying, the eigenvalues λ_1, λ_2 of the transition matrix of the system (5.3.25), $\Phi(\tau_1+T, \tau_1)$, must be less than unity. The eigenvalues λ_1 and λ_2 are plotted in terms of k_1 in Fig. 5.4.1 and Fig.5.4.2. We see that if $0 < k_1 < .1$, then $\lambda_1 < 1$ and $\lambda_2 < 1$, that is r,s will be damped.

We select $k_1 = 4 \times 10^{-4}$ and $k_2 = 8 \times 10^{-4}$.

Implementation of the control law of (5.2.15) with the above k_1 and k_2 requires electric power of about 10 watts.

We numerically integrated the equations (5.3.4) and (5.3.5) with the above k_1 and k_2 for an initial condition of $\psi(0) = 0.3^\circ$, $\dot{\psi}(0) = 0.1^\circ/sec$, $\phi(0) = 0.2^\circ$ $\dot{\phi}(0) = 0.1^\circ/sec$. The roll-yaw motion is plotted in Fig. 5.4.3 and Fig. 5.4.4. With the same numerical values, we also simulated the approximate solution as

given by (5.3.22). The roll-yaw motion by this new approach is also plotted, in Fig. 5.4.5 and Fig. 5.4.6. Comparing Fig. 5.4.3 to Fig. 5.4.5 and Fig. 5.4.4 to Fig. 5.4.6, we found our asymptotic solution to be very accurate. Further, the system is stable in the way we expected and the variations of p, q and r, s can be clearly identified.

5.5 Another Approach: By Generalized Multiple Scales (GMS) Method Using Nonlinear Clocks

It was pointed out by Dr. Ramnath that system (5.3.4) and (5.3.5) can be regarded as a linear system with slowly time-varying coefficients. That is, the coefficients of the equations change at a much slower rate than the dynamic motion of the system [3,4]. This kind of problem, can be easily transformed into a singular perturbation problem by letting $\tau = \varepsilon t$ and changing the independent variable t into τ. Then, another approximate solution by general multiple time scales approach using a set of non-linear time scales is immediately available.

The solution for a second order singularly perturbed system is reviewed in section 2.3 and a similar solution for an n-th order singularly perturbed system is given in [3,4]. To illustrate, equations (5.3.4') and (5.3.5')

can be decoupled and ϕ, ψ satisfy an equation of the form:

$$\sum_{i=0}^{4} a_i(\varepsilon t)\, \alpha^{(i)} = 0$$

The GMS solution for this equation is [3,4]:

$$\tilde{\alpha}(t,\varepsilon) = \sum_{i=1}^{4} C_i \left(\frac{\partial F}{\partial k_i}\right)^{-\frac{1}{2}} \exp\left(\tau_{1i}(t,\varepsilon)\right)$$

where
$$F(k,\tau_0) \equiv \sum_{i=0}^{4} a_i(\tau_0)\, k^i \ ;$$

$$\tau_{1i} = \int \frac{k_i}{\varepsilon}\, dt \quad \text{and } k_i \text{ satisfy } F = 0$$

This GMS solution employs nonlinear scales τ_{1i} in contrast with the linear scales of the MTS solution. The GMS approach subsumes the MTS method and could lead to more accurate description of the dynamics.

The advantages of this alternative approach are twofold. First, it treats the whole class of problems of linear slowly time-varying systems and thereby it can, conceivably, deal with a more complicated problem. Second, it is fairly easy to numerically implement this approximate solution which needs much less computer time than straight direct integration. Thus it might be helpful in the area of simulating a system if the design task has to be carried out by trial and error.

The same numerical example as discussed in the previous section is used. This time, the solution is approximated by the GMS method. The result is given by Fig. 5.5.1 for roll motion and Fig. 5.5.2 for yaw motion. The accuracy is found to be excellent. Also for demonstrating that the new approach can save computer time, several cases have been tried, and the results are summarized in Table 5.5.

CHAPTER 6

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

6.1 Conclusions

A general method has been given for fast prediction of a satellite attitude motion under the influence of its environment disturbances. An approximate solution to the problem is developed by using the multiple time scale asymptotic approach, such that the digital implementation of these approximations would give a significant saving of computer time as compared to direct simulation. Furthermore, because the new approach has been designed to be very general it can handle any orbit, initial condition or satellite mass distribution, and so it could potentially become a valuable tool in satellite engineering.

The attitude motion of a rigid body asymmetric satellite is first considered. By the MTS asymptotic technique, the slow secular and the fast oscillatory effects of the disturbing torque on the attitude motion can be immediately separated and then be evaluated individually. These slow and fast behaviors, combined, give the complete motion while divided, each describes a different aspect of the phenomenon.

Similarly, a class of dual spin satellites is then studied. A dual spin satellite represents a vehicle

carrying sizable fly-wheels on-board. This model may resemble many satellites in use today, since the fly-wheel has been a common device for attitude control and stabilization. We have studied a special case of dual spin satellite with a single fly-wheel mounted along one of the vehicle body principal axes. However, the problem of a general dual spin satellite with multiple fly-wheels mounted in different directions seems still to require further research.

The new approach is then numerically simulated for two environment disturbances. One is a satellite disturbed by the gravity gradient torque and the other is by geomagnetic torque. The results show that the new method has a significant advantage over the conventional direct integration. In some situations it can be faster by an order of magnitude while the approximation errors are still well bounded and acceptable.

A way of handling resonant situations is also discussed. Attitude resonance will occur if the satellite has a mass distribution such that a low-order commensurability exists between the polhode frequency and the angular velocity frequency. Then there will be a substantial increase in the secular effect due to the disturbance. We found that the resonant situation can be

easily detected and handled in our approach.

Finally, as the MTS formulation separates the slow and fast behaviors of a satellite attitude motion, we use this property for the design of an attitude control device. In the problem, the control torque from geomagnetic interaction is very small. Nevertheless, its influence on the dynamics becomes clear if we look at the secular behavior on a slow clock. This idea has also been explored in [41]. However, we believe that the complete solution to the problem is achieved for the first time and the control law is new.

6.2 Suggestions for Future Research

This research concerns the attitude motion of a satellite which is operated in a passive mode. It is therefore essential for us to consider all the possible major disturbances. We know, besides the gravity gradient and geomagnetic torques, there are also other disturbances which are important in particular situations, as for instance, the atmospheric drag in a low orbit case and solar radiation pressure for a satellite with large surface area. The latter is found to be important and worthy of research because more and more satellites, especially the long-lived ones, have panels to collect the

sun's radiation for their energy supply. This problem, however, is a difficult one, since the motion of the earth around the sun, which changes the direction of the sun light, represents an even slower mode on top of the fast attitude rotation and the slow satellite orbital motion. Furthermore, the earth shadow could give a discontinous change of the radiation disturbing torque.

The second area is involved with the generalization of the prediction method for all dual spin satellites; that is, satellites which contain multiple fly-wheels. The difficulties of the problem have been listed in chapter 4. We believe that by further research, these difficulties could be overcome.

In chapter 5, we have applied an asymptotic technique for a classic control design problem. The asymptotic approach seems very helpful in that particular case. We believe that a similar situation, whenever the control force is small, could also occur in other fields such as orbit transfer by a low thrust engine. Therefore a systematic study of the role of asymptotic methods in control theory could be very interesting and useful.

APPENDIX A

EXPRESSION FOR $\overset{*}{Q}(\overset{*}{I}_m^{\,-1}\ \overset{*}{Q}^{-1}\bar{v}x)\ \overset{*}{Q}^{-1}\bar{v}$

If $\overset{*}{I}_m$ is the moment of inertia, then for arbitrary matrix $\overset{*}{A}$ and vector $\bar{v}$, the expression of $(\overset{*}{A}\bar{v}x)\ \overset{*}{I}_m\overset{*}{A}\bar{v}$ can be re-grouped as follows:

$$
\begin{aligned}
&(\overset{*}{A}\bar{V}x)\ \overset{*}{I}_m\ \overset{*}{A}\ \bar{V} \\
&= \begin{bmatrix} I_z - I_y & 0 & 0 \\ 0 & I_x - I_z & 0 \\ 0 & 0 & I_y - I_x \end{bmatrix}
\left\{ \begin{bmatrix} A_{21}A_{31} & A_{21}A_{32} + A_{22}A_{31} & A_{21}A_{33} + A_{23}A_{31} \\ A_{11}A_{31} & A_{11}A_{32} + A_{12}A_{31} & A_{11}A_{33} + A_{13}A_{31} \\ A_{11}A_{21} & A_{11}A_{22} + A_{12}A_{21} & A_{11}A_{23} + A_{13}A_{21} \end{bmatrix} v_1 \right. \\
&\quad \left. + \begin{bmatrix} 0 & A_{22}A_{32} & A_{22}A_{33} + A_{23}A_{32} \\ 0 & A_{12}A_{32} & A_{12}A_{33} + A_{13}A_{32} \\ 0 & A_{12}A_{22} & A_{12}A_{23} + A_{13}A_{22} \end{bmatrix} v_2
+ \begin{bmatrix} 0 & 0 & A_{23}A_{33} \\ 0 & 0 & A_{13}A_{33} \\ 0 & 0 & A_{13}A_{23} \end{bmatrix} v_3 \right\} \bar{V}
\end{aligned}
$$

$$
= \overset{*}{I}_s \left[\overset{*}{OP}_1(\overset{*}{A})\,v_1 + \overset{*}{OP}_2(\overset{*}{A})\,v_2 + \overset{*}{OP}_3(\overset{*}{A})\,v_3 \right] \bar{V} \qquad \text{(A.1)}
$$

where $\overset{*}{I}_s$ and $\overset{*}{OP}_i(\overset{*}{A})$, i = 1.2.3 are defined by comparing the above two expressions and v_i, i = 1,2,3 are three components of the vecror $\bar{v}$. With these definitions,

$$\overset{*}{Q}(\overset{*}{I}{}_m^{-1}\overset{*}{Q}{}^{-1}\bar{v}\times)\overset{*}{Q}{}^{-1}\bar{v}$$

$$= \overset{*}{Q}\left[(\overset{*}{I}{}_m^{-1}\overset{*}{Q}{}^{-1})\bar{v}\times\right]\overset{*}{I}_m(\overset{*}{I}{}_m^{-1}\overset{*}{Q}{}^{-1})\bar{v}$$

$$= \overset{*}{Q}\overset{*}{I}_s\left[\overset{*}{OP}_1(\overset{*}{I}{}_m^{-1}\overset{*}{Q}{}^{-1})v_1 + \overset{*}{OP}_2(\overset{*}{I}{}_m^{-1}\overset{*}{Q}{}^{-1})v_2\right.$$

$$\left. + \overset{*}{OP}_3(\overset{*}{I}{}_m^{-1}\overset{*}{Q}{}^{-1})v_3\right]\bar{v} \tag{A.2}$$

In case if $\overset{*}{Q}$ is a periodic matrix, we can expand $\overset{*}{Q}\,\overset{*}{I}_s\,\overset{*}{OP}_i\,(\overset{*}{I}{}_m^{-1}\overset{*}{Q}{}^{-1})$ into a Fourier series; that is

$$\overset{*}{Q}(\overset{*}{I}{}_m^{-1}\overset{*}{Q}{}^{-1}\bar{v}\times)\overset{*}{Q}{}^{-1}\bar{v}$$

$$= \sum_{i=1}^{3}\left\{\overset{*}{E}_{i0} + \sum_{j=1}^{n}\left[\overset{*}{E}_{ij}\sin\left(\frac{2\pi j t}{T\omega}\right) + \overset{*}{F}_{ij}\cos\left(\frac{2\pi j t}{T\omega}\right)\right]\right\} v_i\,\bar{v} \tag{A.3}$$

APPENDIX B

EXPANSION OF $\overset{*}{\Phi}_{\beta}^{-1} [\overset{*}{\delta\omega}] \overset{*}{\Phi}_{\beta} \bar{\beta}_{1N}$

The transition matrix $\overset{*}{\Phi}_{\beta}(t,o)$ has been given by (3.3.29); if we substitute that into the expression of $\overset{*}{\Phi}_{\beta}^{-1} [\overset{*}{\delta\omega}] \overset{*}{\Phi}_{\beta} \bar{\beta}_{IN}$, we will have

$$\overset{*}{\Phi}_{\beta}^{-1} [\overset{*}{\delta\omega}] \overset{*}{\Phi}_{3} \bar{\beta}_{IN}$$

$$= \left\{ \begin{pmatrix} 0 , & -\frac{E_2}{E_1}\cos(P_1-P_2) , & 0 , & \frac{E_2}{E_1}\sin(P_1-P_2) \\ \frac{E_1}{E_2}\cos(P_1-P_2) , & 0 , & \frac{E_1}{E_2}\sin(P_1-P_2) , & 0 \\ 0 , & \frac{E_2}{E_1}\sin(P_1-P_2) , & 0 , & \frac{E_2}{E_1}\cos(P_1-P_2) \\ -\frac{E_1}{E_2}\sin(P_1-P_2) , & 0 , & \frac{E_1}{E_2}\cos(P_1-P_2) , & 0 \end{pmatrix} \right.$$

$$\left. \cdot \begin{pmatrix} \delta\omega_1 & 0 & \delta\omega_3 & 0 \\ 0 & \delta\omega_1 & 0 & \delta\omega_3 \\ \delta\omega_3 & 0 & -\delta\omega_1 & 0 \\ 0 & -\delta\omega_3 & 0 & \delta\omega_1 \end{pmatrix} + \begin{bmatrix} 0 & 0 & -\delta\omega_2 & 0 \\ 0 & 0 & 0 & -\delta\omega_2 \\ \delta\omega_2 & 0 & 0 & 0 \\ 0 & \delta\omega_2 & 0 & 0 \end{bmatrix} \right\} \bar{\beta}_{IN} \tag{B.1}$$

where E_1, E_2, P_1, P_2 are defined in Section 3.3. They are all periodic functions with period of T_ω.

Substitute $\delta\bar{\omega}$ from (3.5.30), and expand all the periodic matrices by Fourier series. It is tedious but straightforward to have $\overset{*}{\Phi}_{\beta}^{-1} [\delta\overset{*}{\omega}] \overset{*}{\Phi}_{\beta} \bar{\beta}_{1N}(\tau_1)$ written as

$$\overset{*}{\Phi}_{\beta}^{-1} [\delta\overset{*}{\omega}] \overset{*}{\Phi}_{\beta} \bar{\beta}_{1N}(\tau_1)$$

$$= \bar{R}_1(\tau_1) + \overset{*}{P}_B(\tau_0) \bar{R}_2(\tau_1) \tag{B.2}$$

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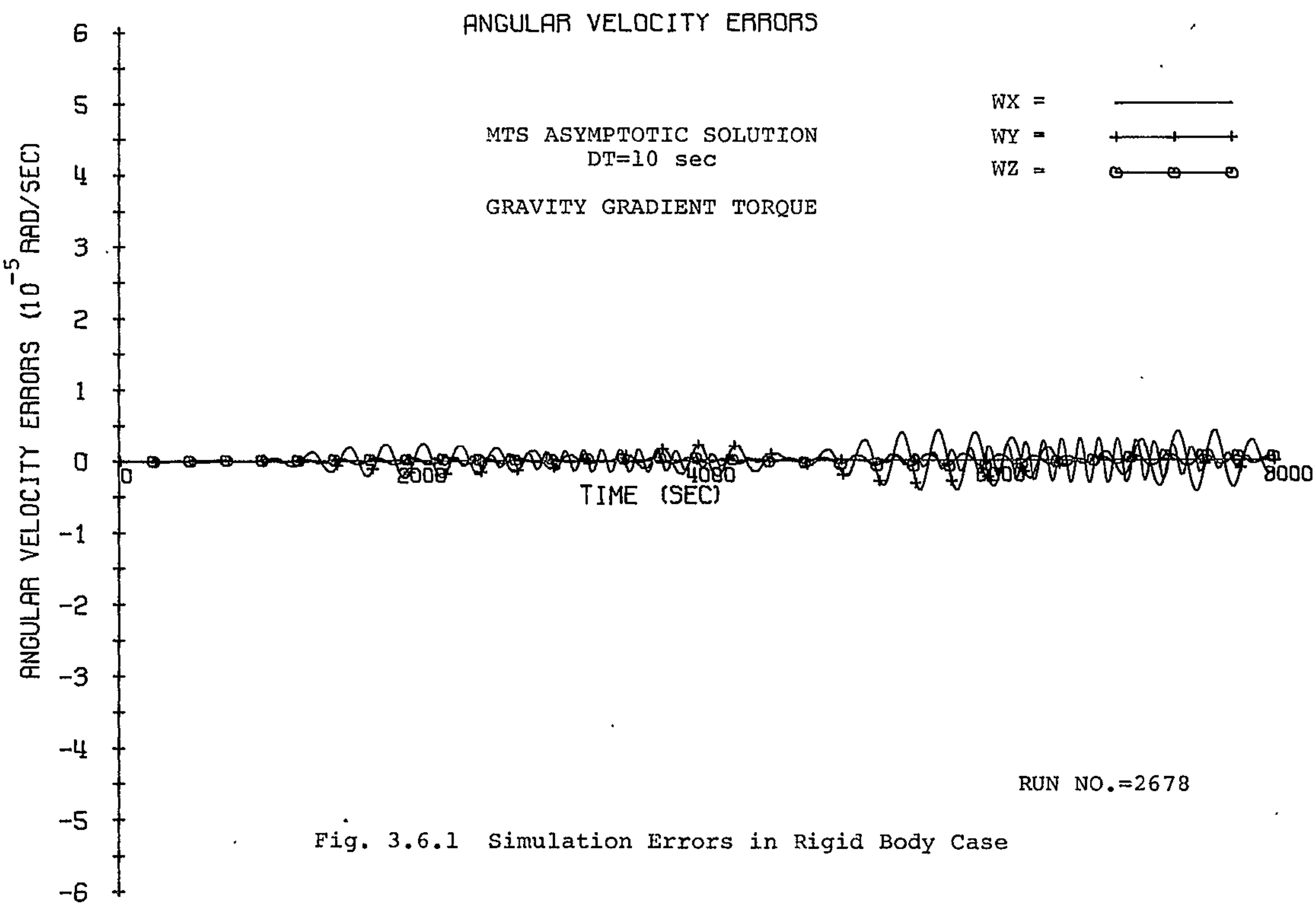


Fig. 3.6.1 Simulation Errors in Rigid Body Case

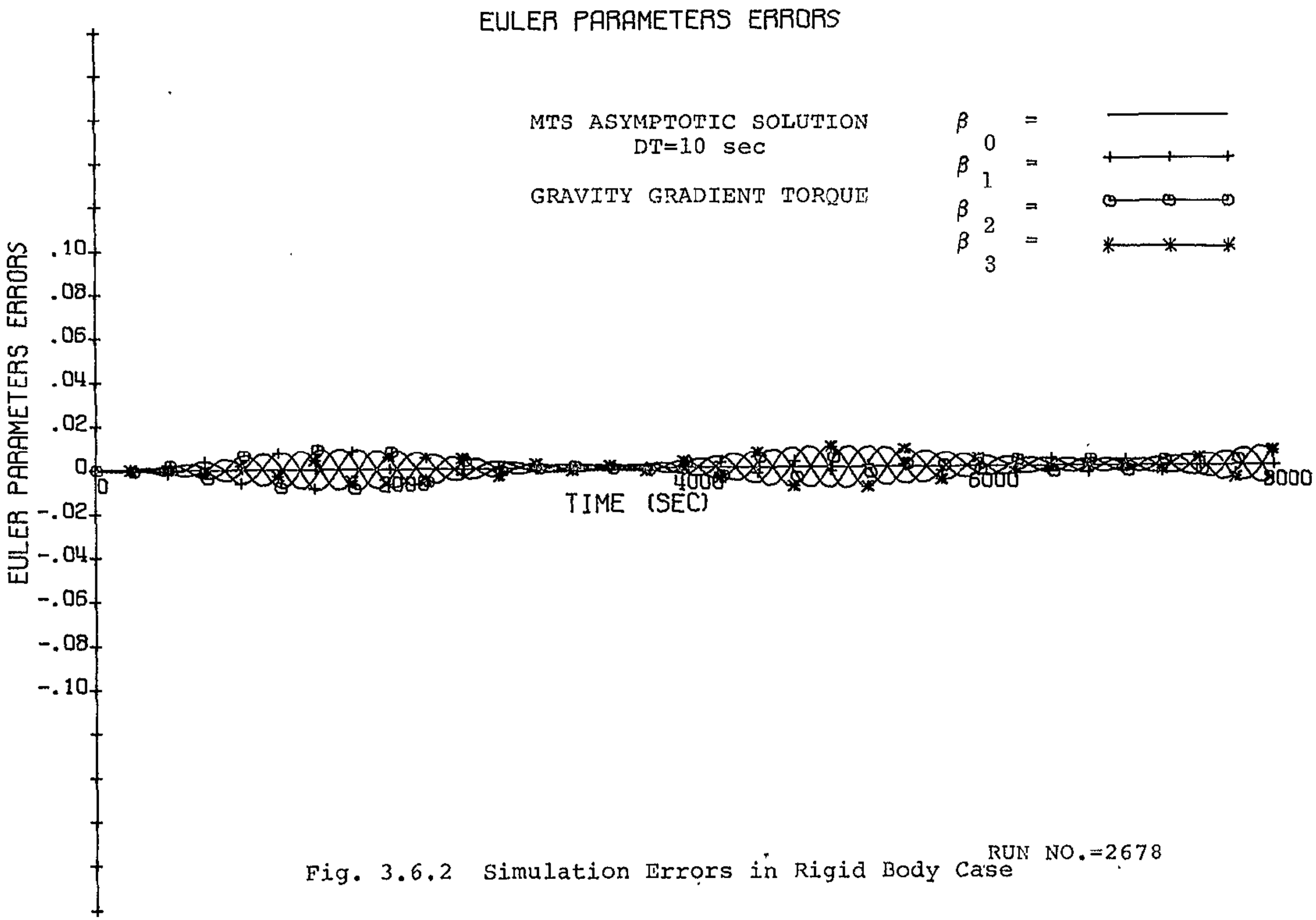


Fig. 3.6.2 Simulation Errors in Rigid Body Case

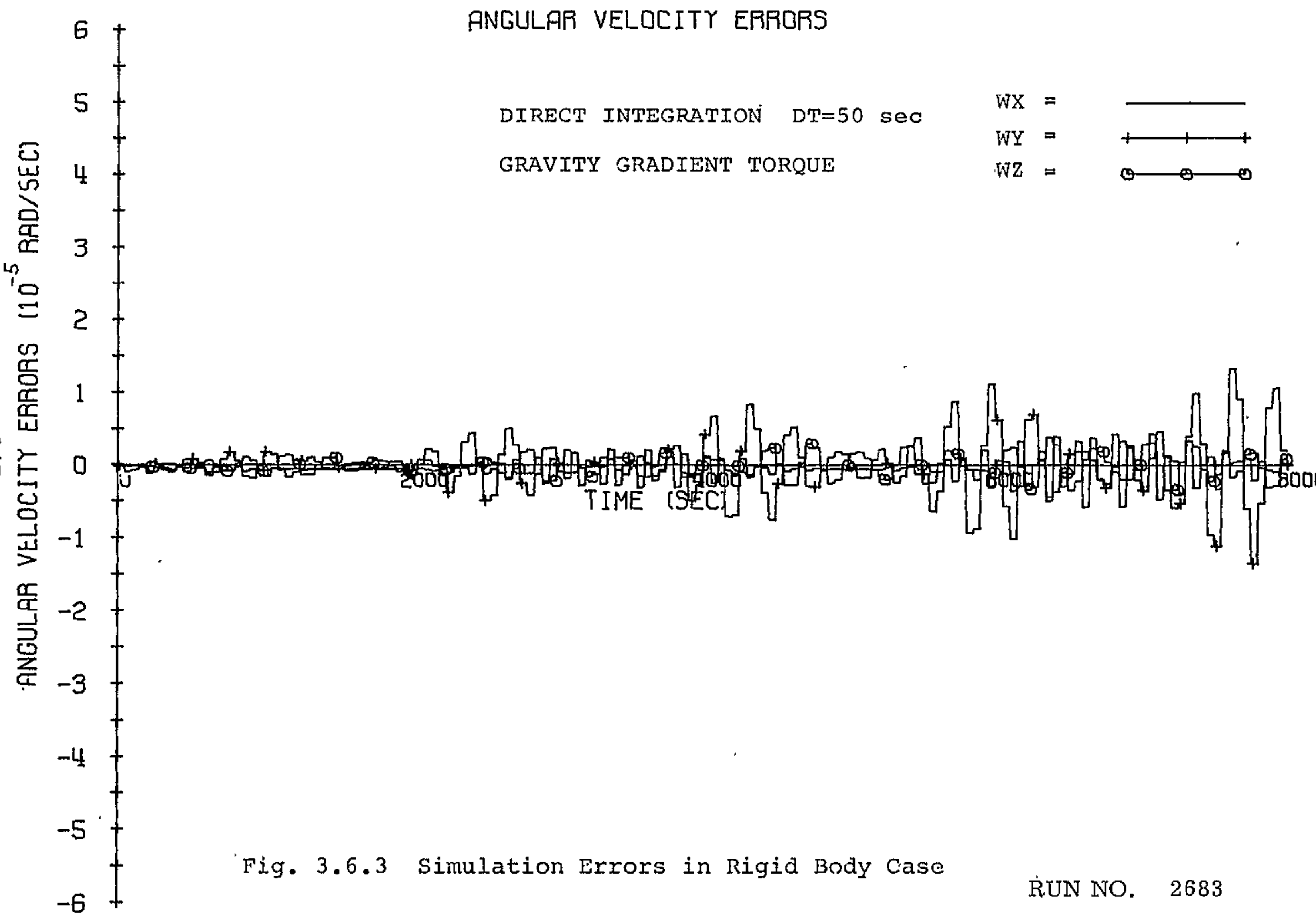


Fig. 3.6.3 Simulation Errors in Rigid Body Case

RUN NO. 2683

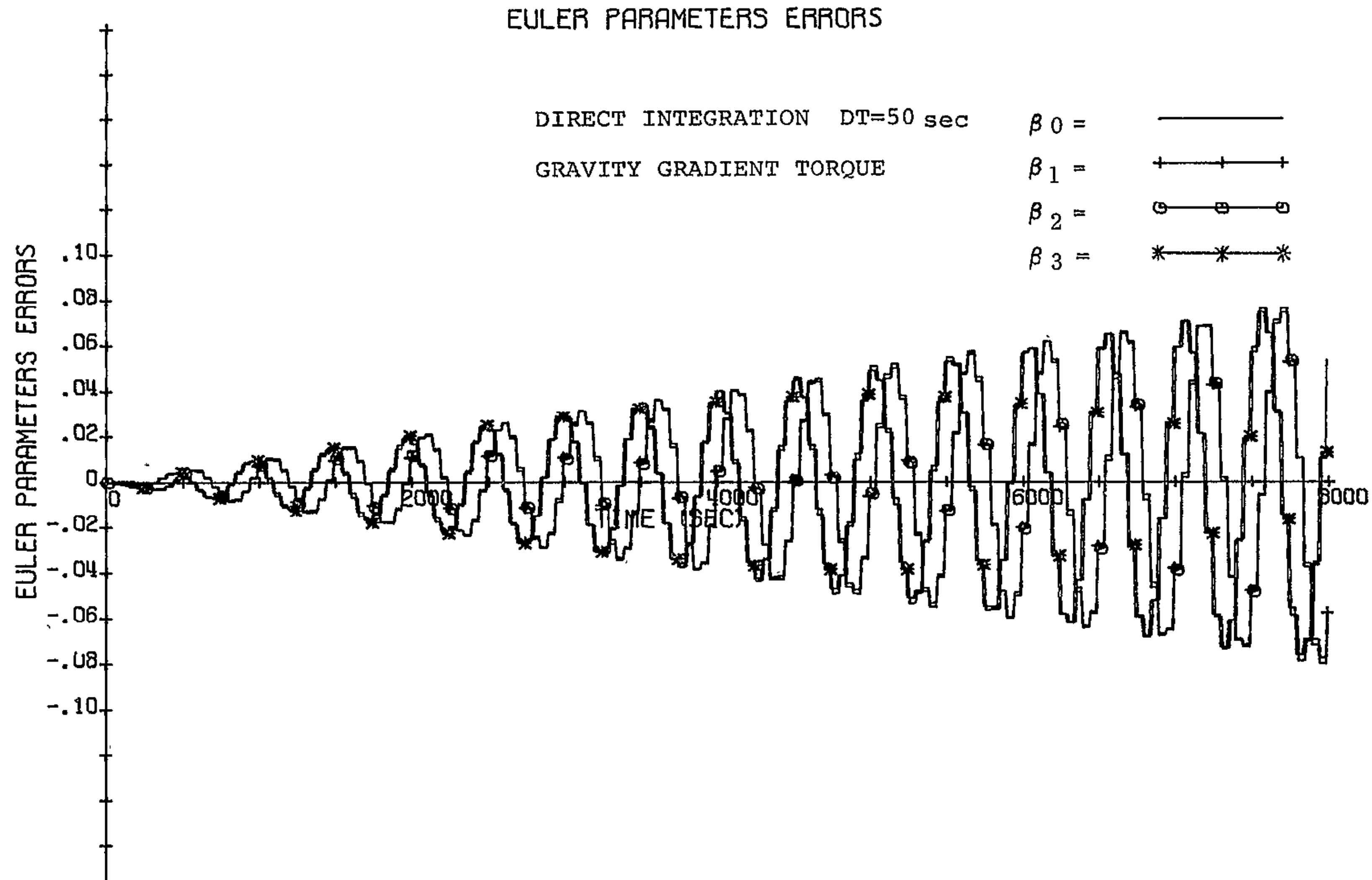


Fig. 3.6.4 Simulation Errors in Rigid Body Case

RUN NO. =2683

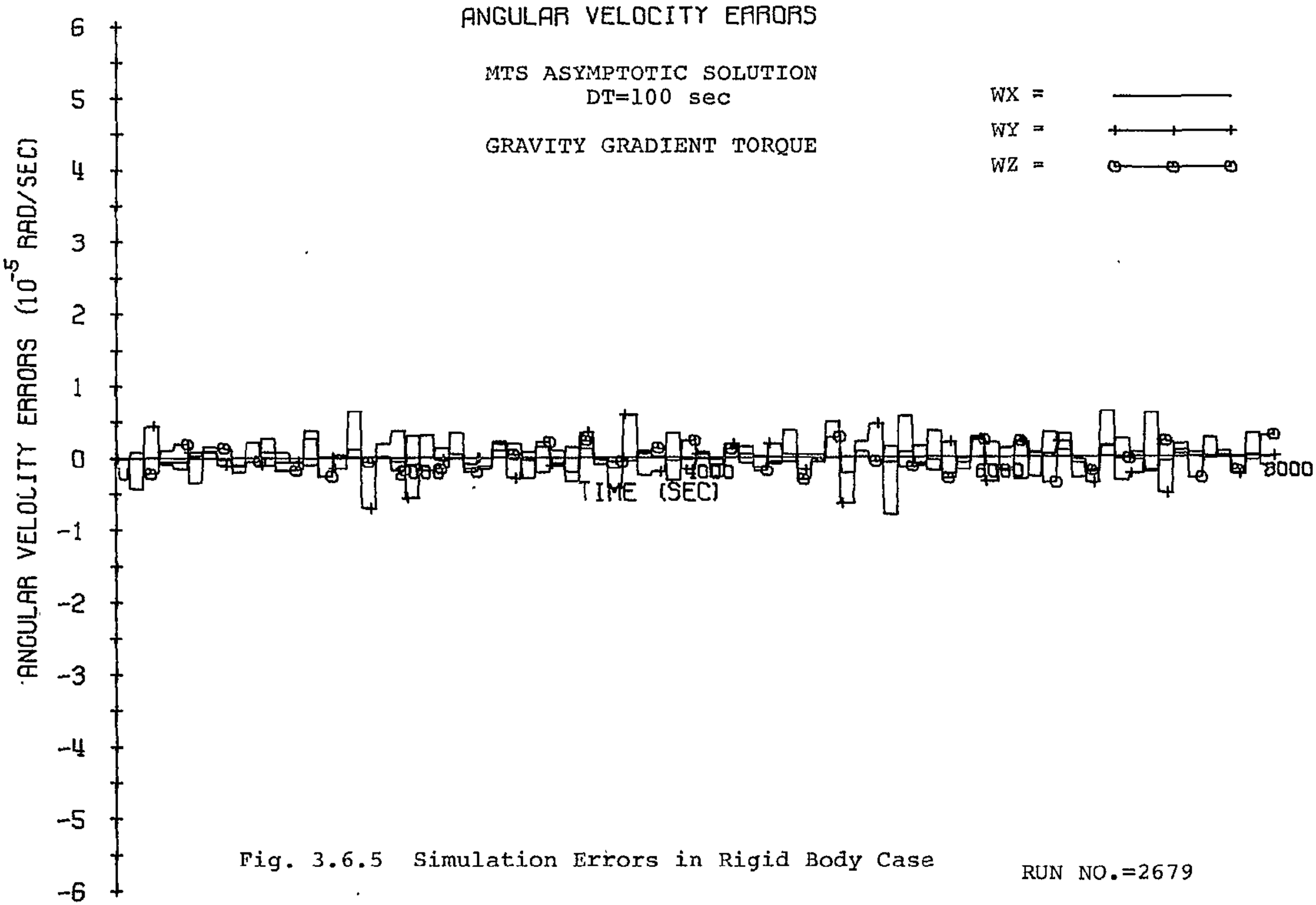


Fig. 3.6.5 Simulation Errors in Rigid Body Case

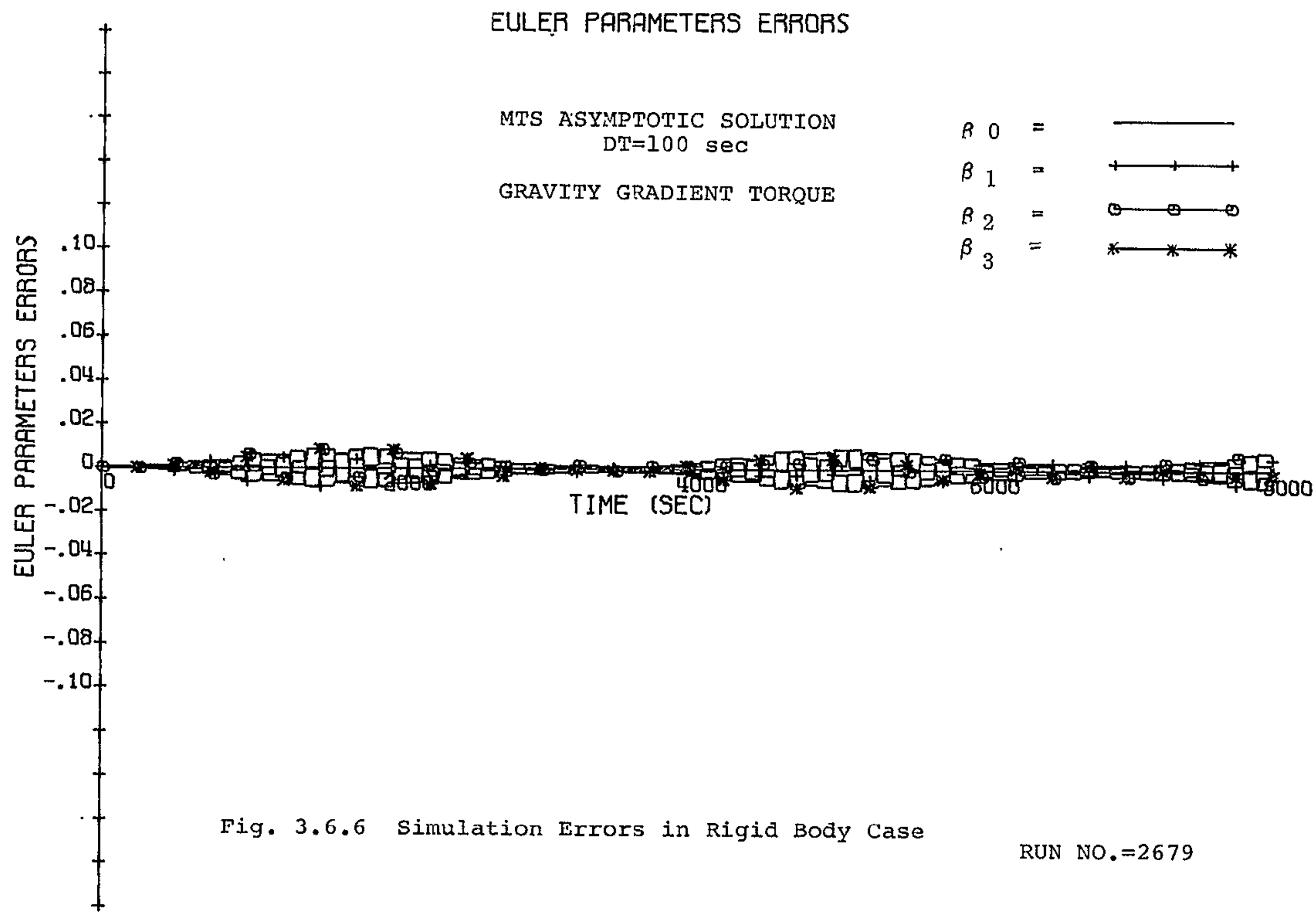


Fig. 3.6.6 Simulation Errors in Rigid Body Case

RUN NO.=2679

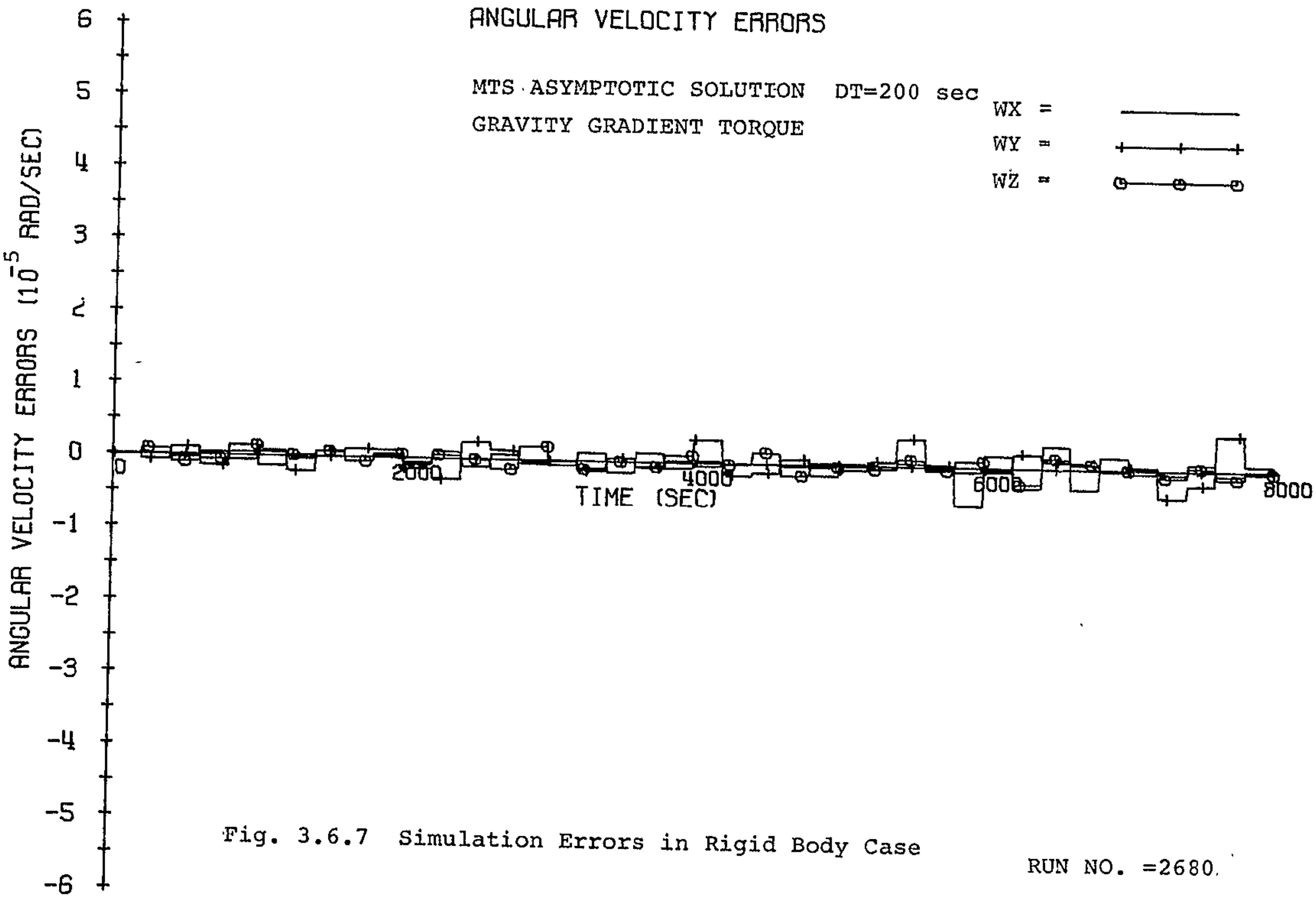


Fig. 3.6.7 Simulation Errors in Rigid Body Case

RUN NO. =2680

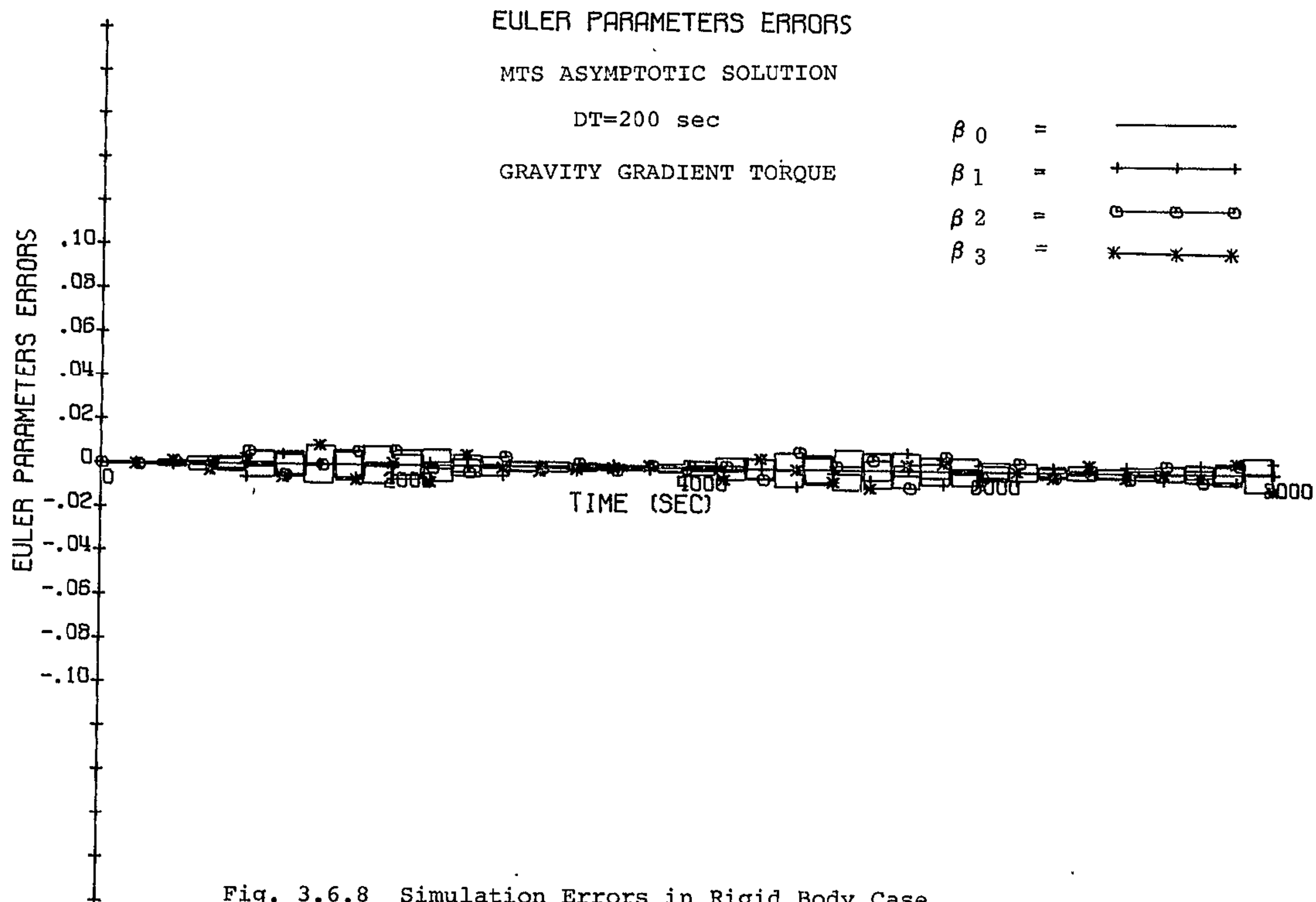


Fig. 3.6.8 Simulation Errors in Rigid Body Case

RUN NO. 2680

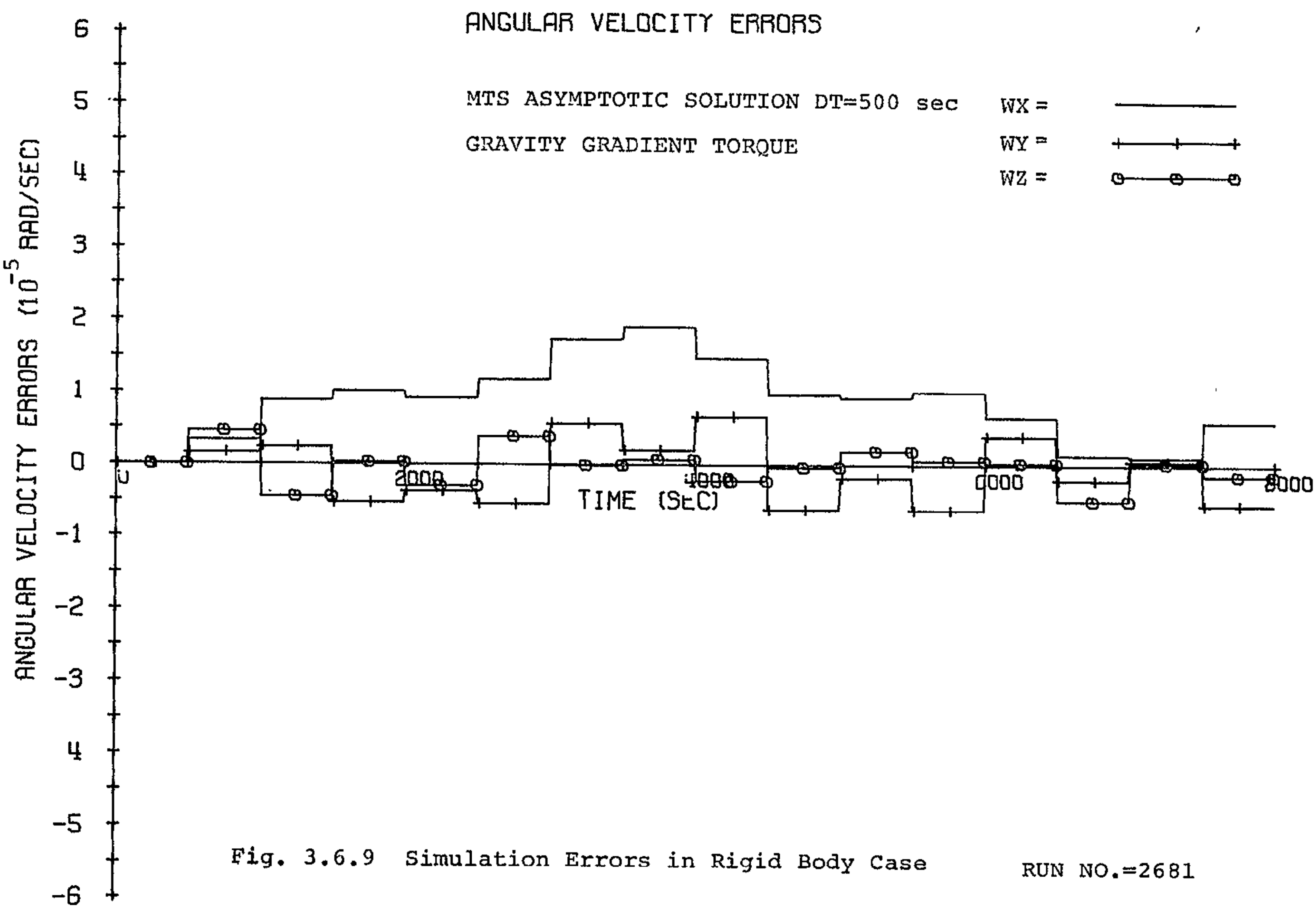


Fig. 3.6.9 Simulation Errors in Rigid Body Case

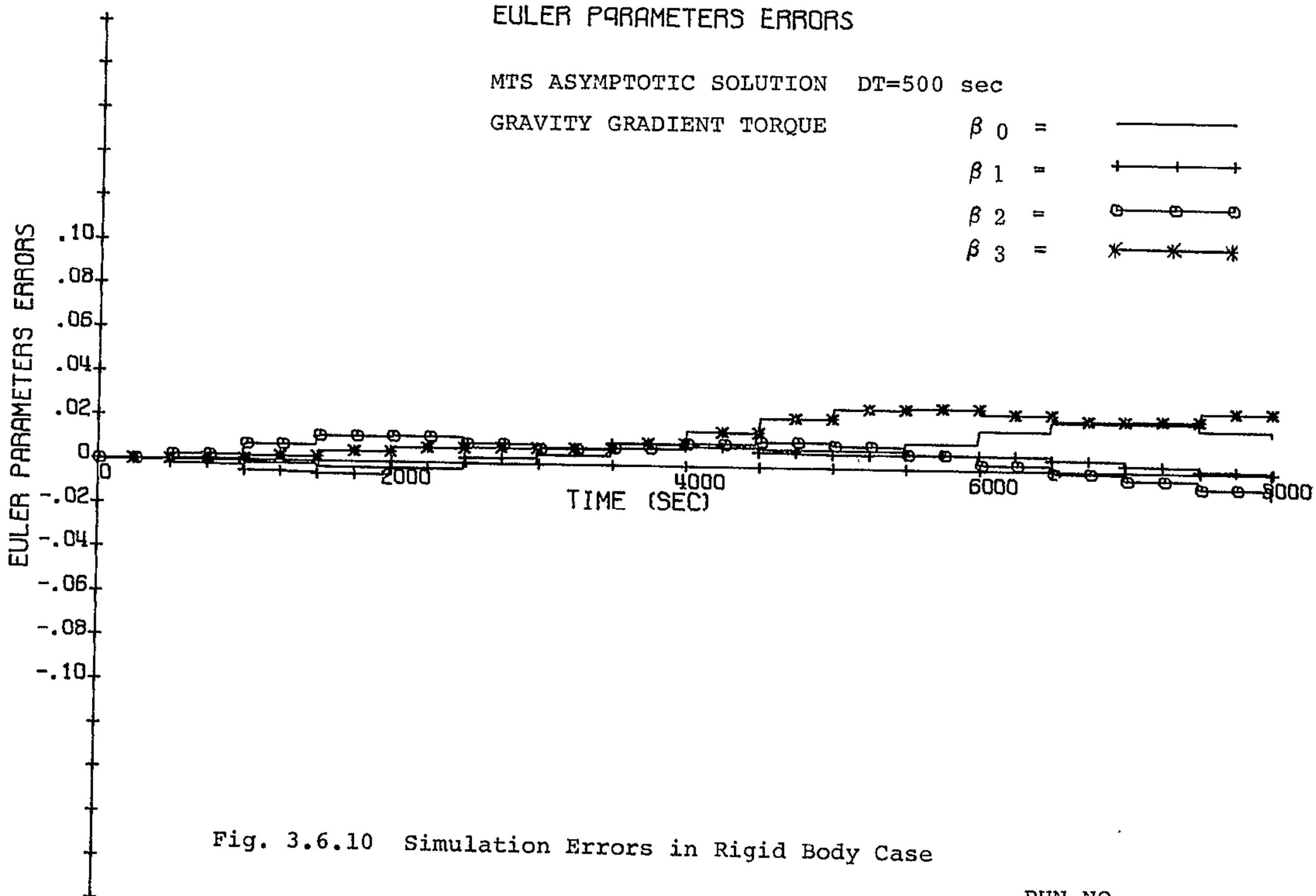


Fig. 3.6.10 Simulation Errors in Rigid Body Case

RUN NO. =2681

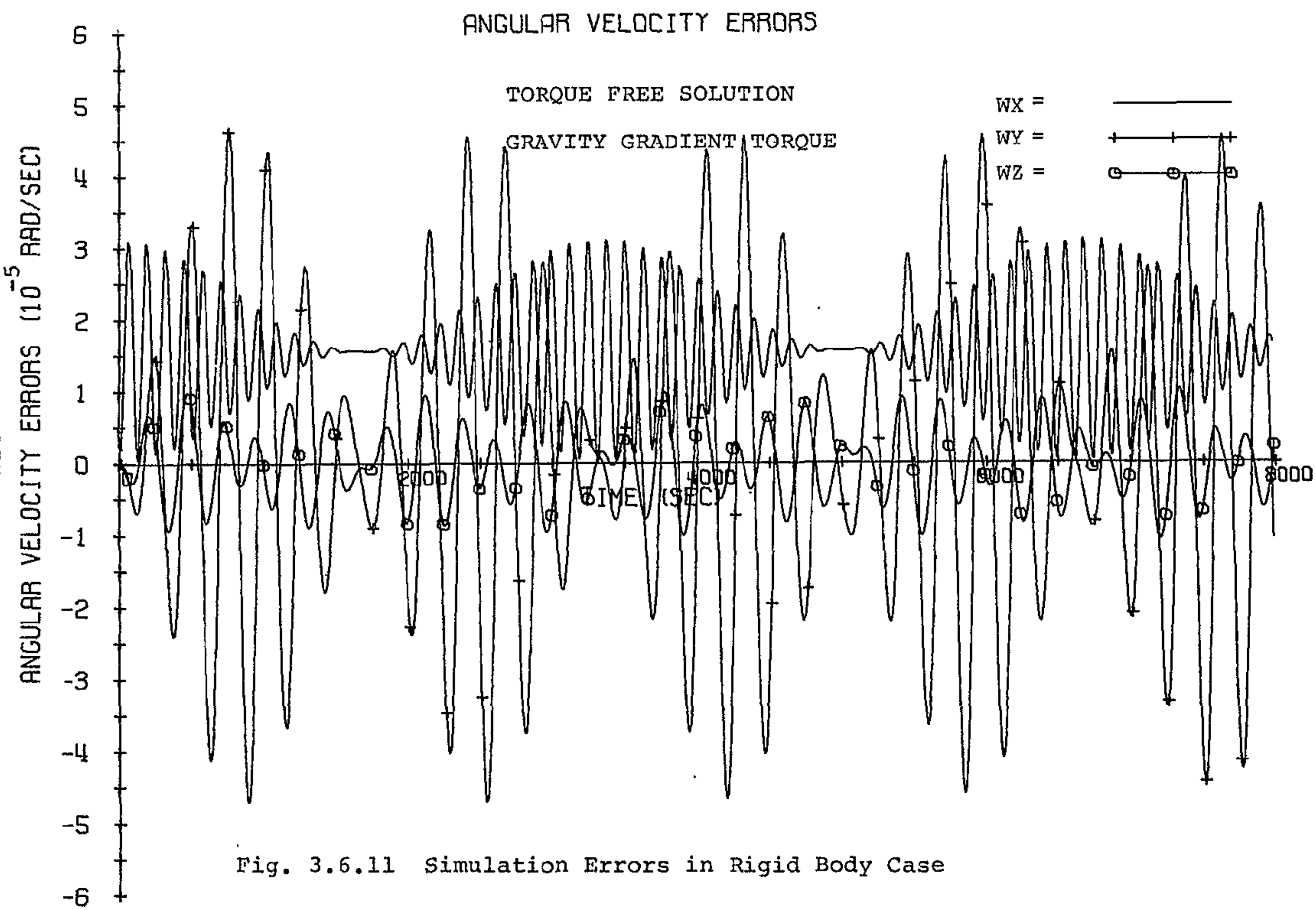


Fig. 3.6.11 Simulation Errors in Rigid Body Case

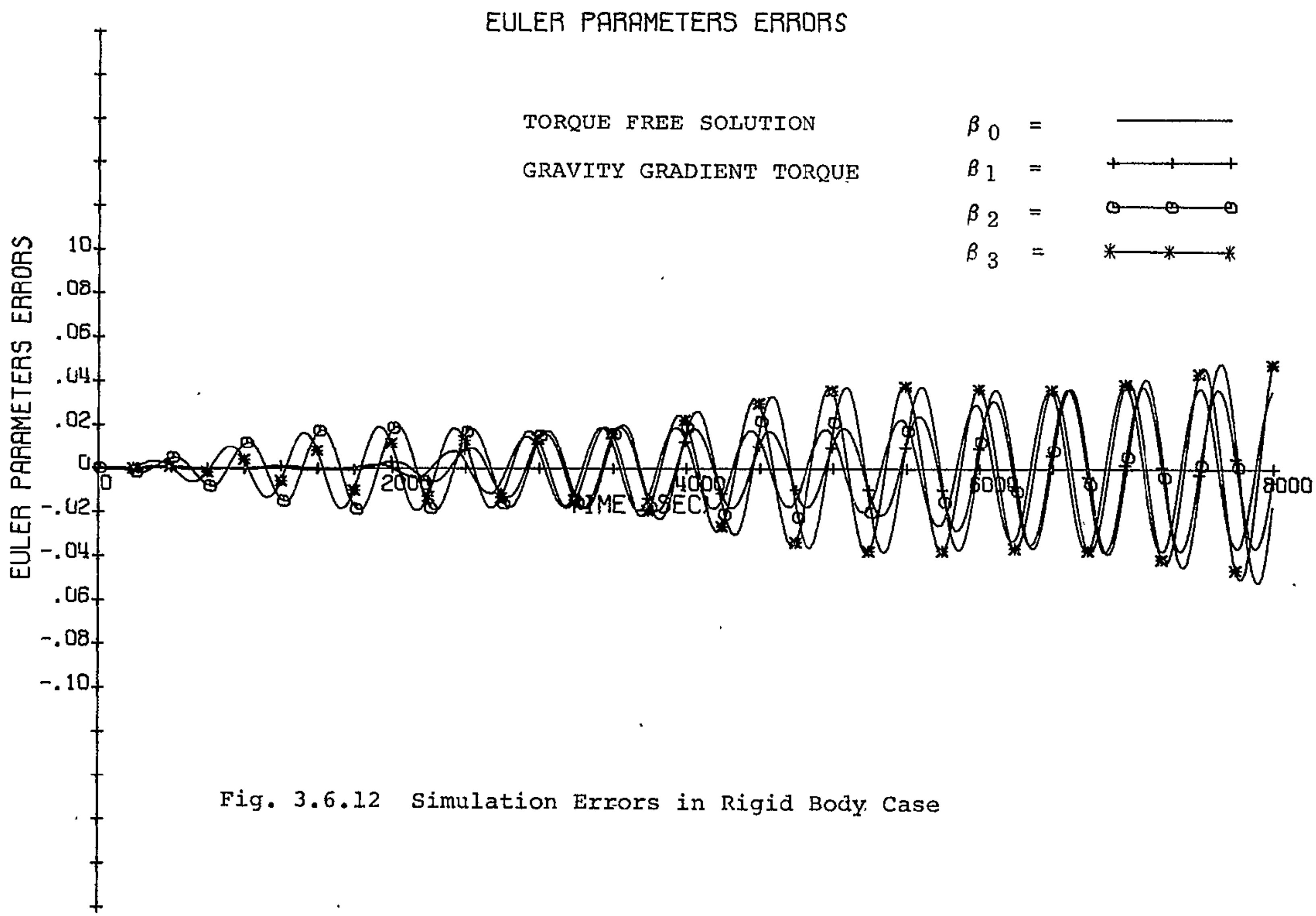


Fig. 3.6.12 Simulation Errors in Rigid Body Case

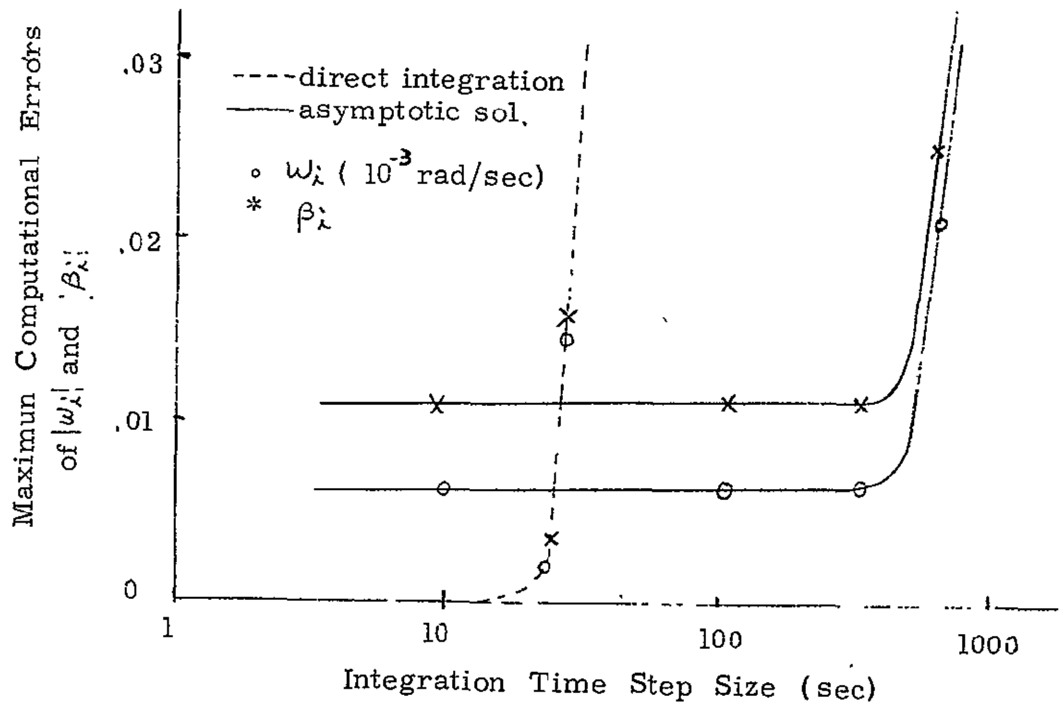


Fig. 3.6.13 Maximum Simulation Errors

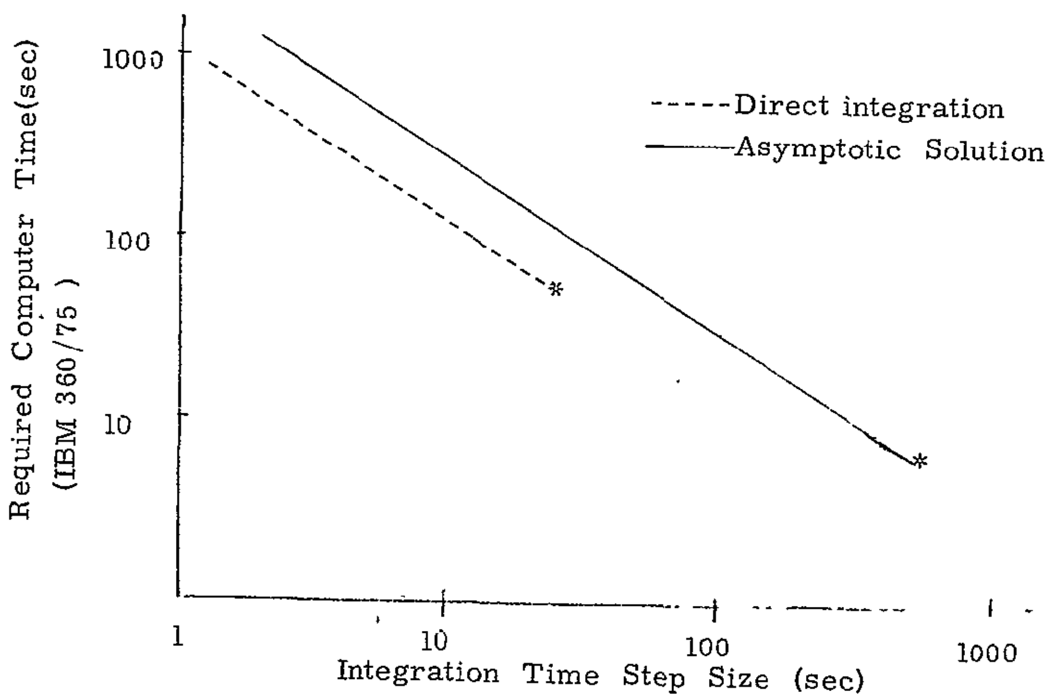


Fig. 3.6.14 Computer Time

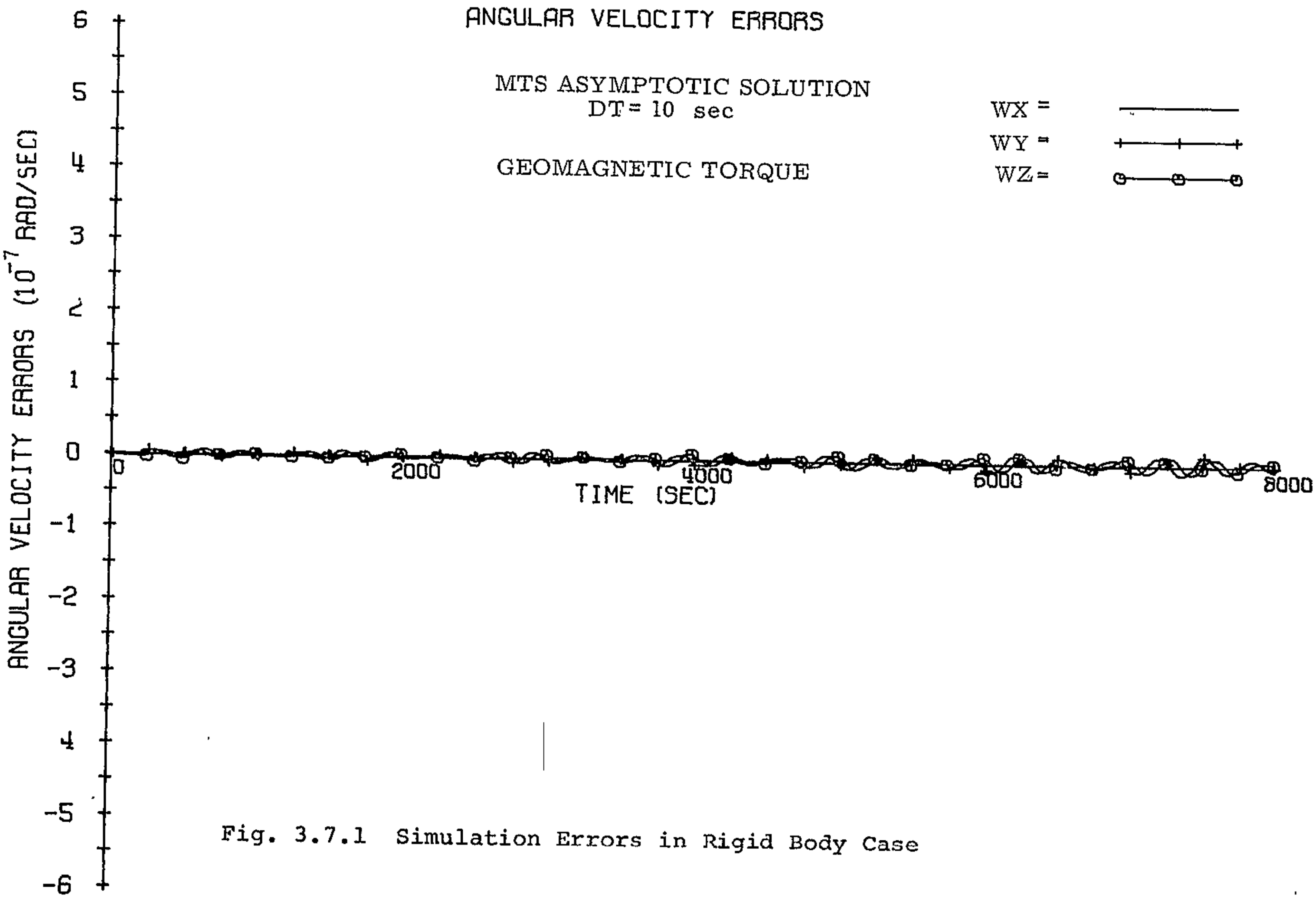


Fig. 3.7.1 Simulation Errors in Rigid Body Case

RUN NO. 2684

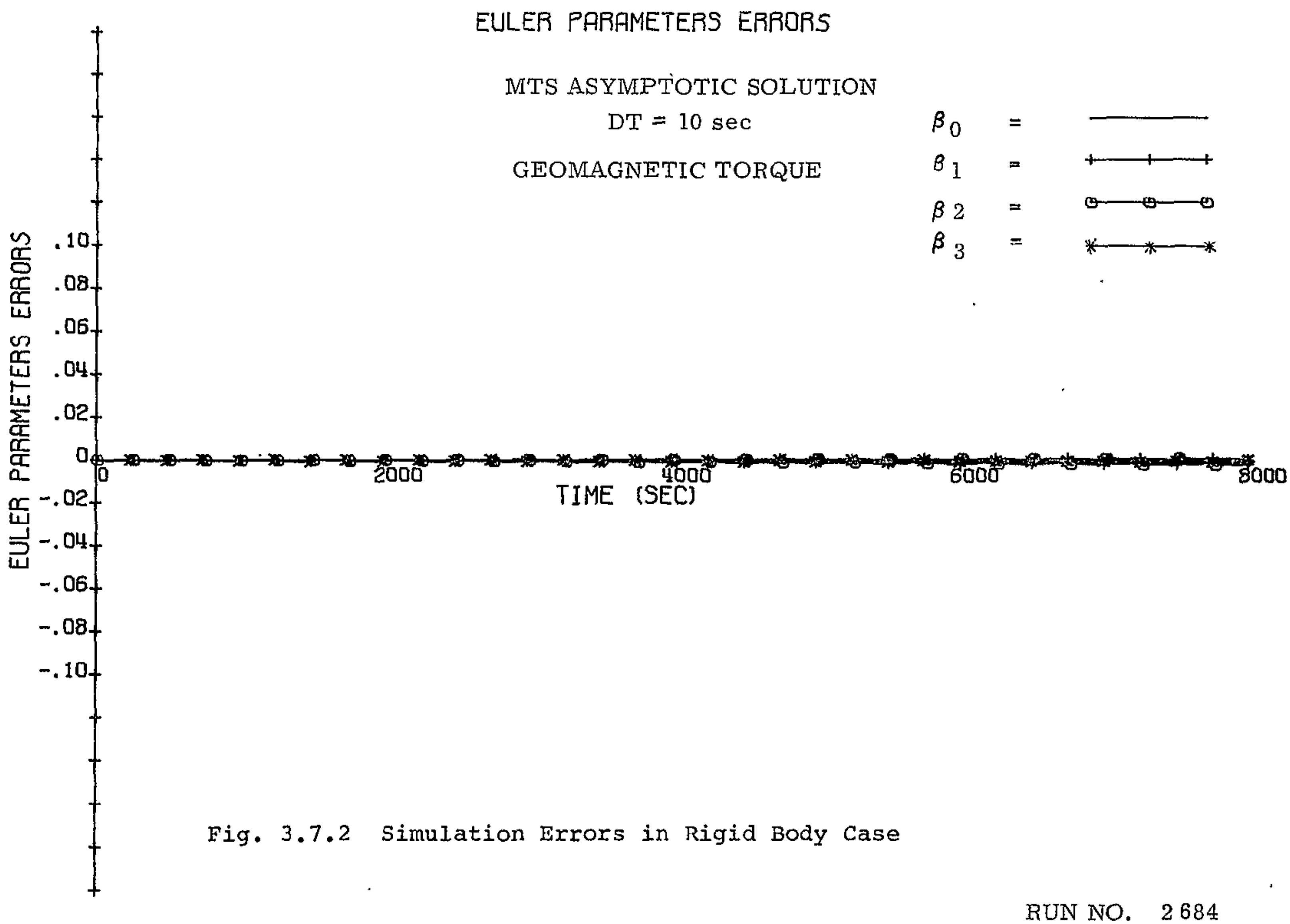


Fig. 3.7.2 Simulation Errors in Rigid Body Case

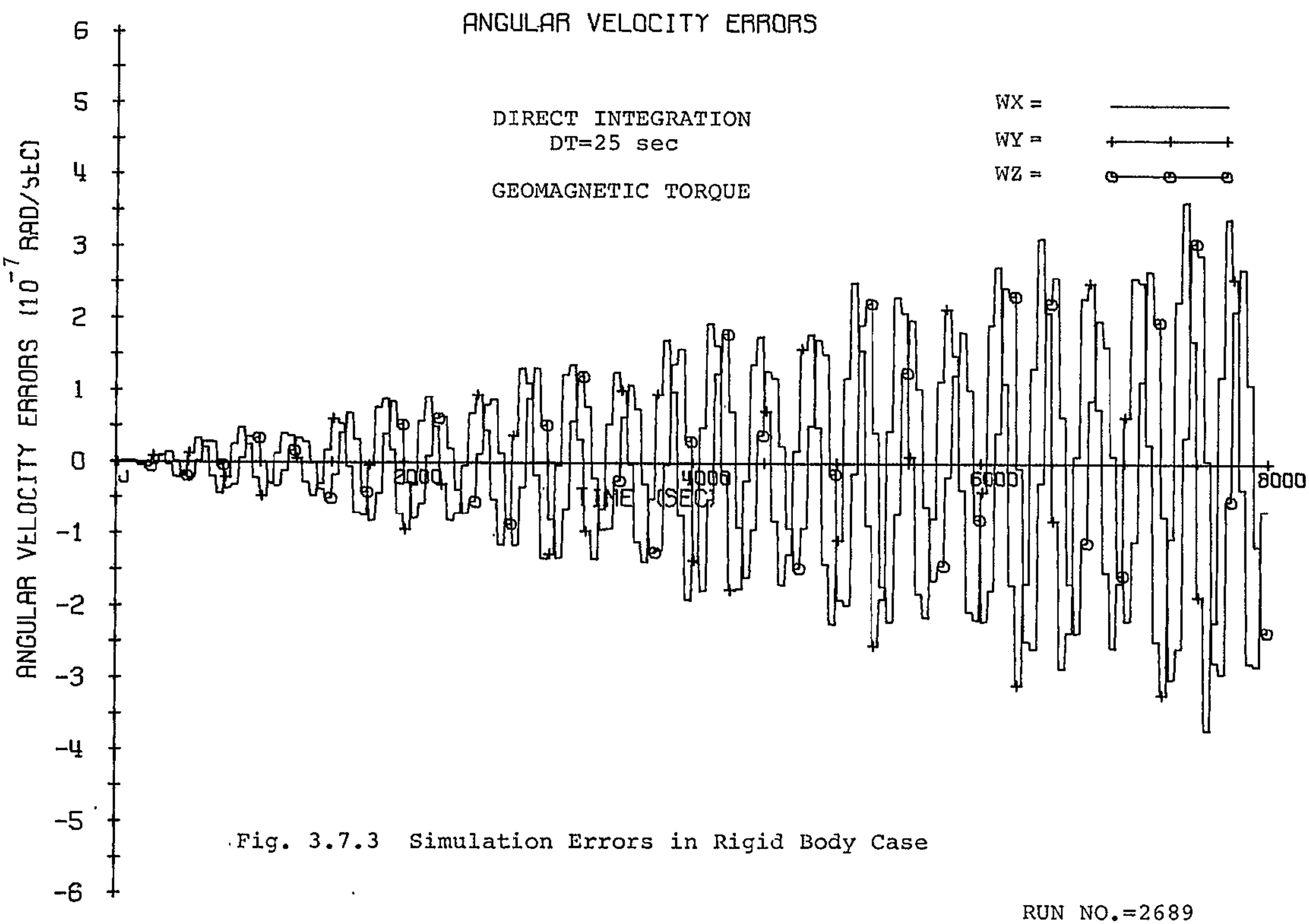


Fig. 3.7.3 Simulation Errors in Rigid Body Case

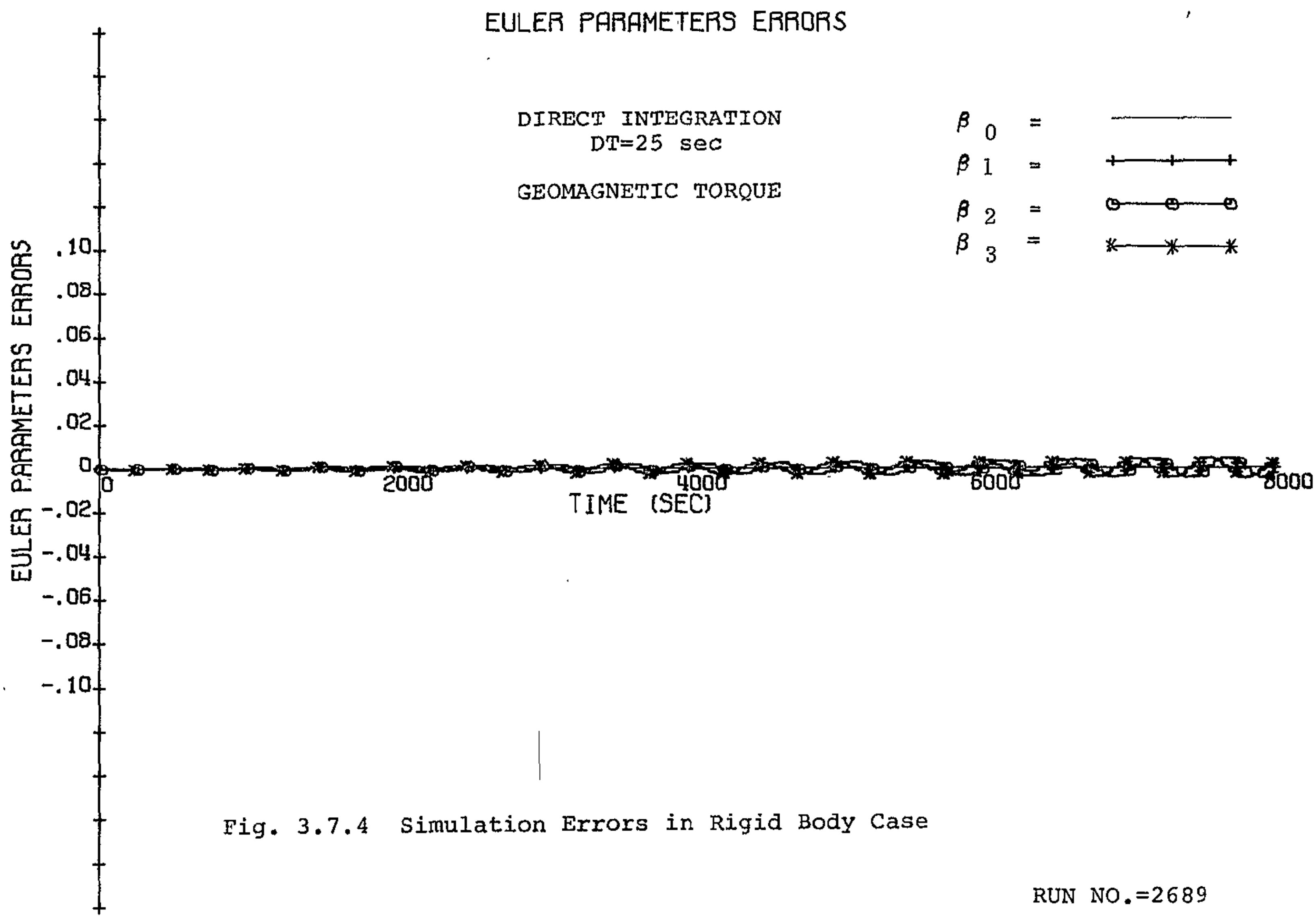


Fig. 3.7.4 Simulation Errors in Rigid Body Case

RUN NO.=2689

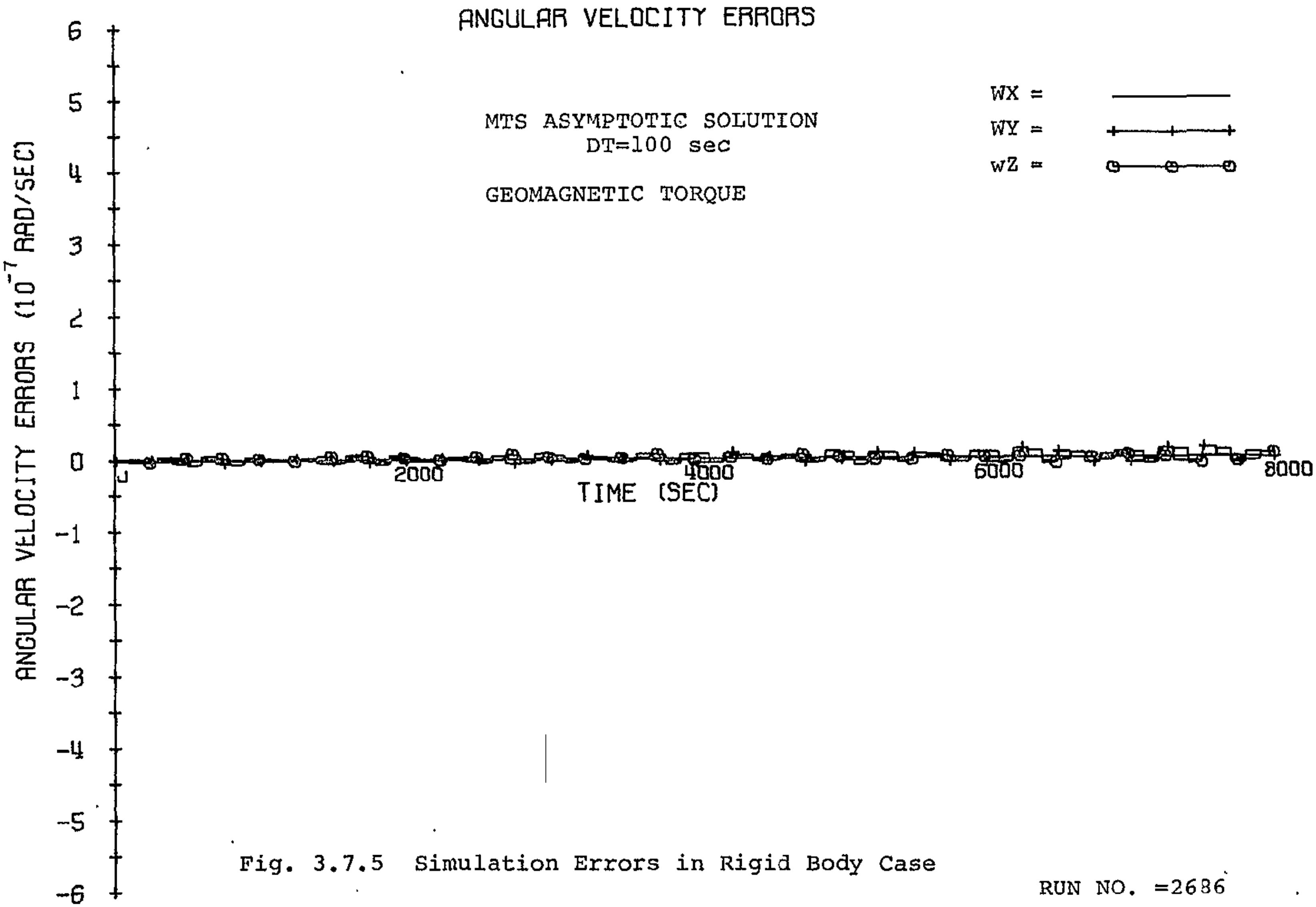


Fig. 3.7.5 Simulation Errors in Rigid Body Case

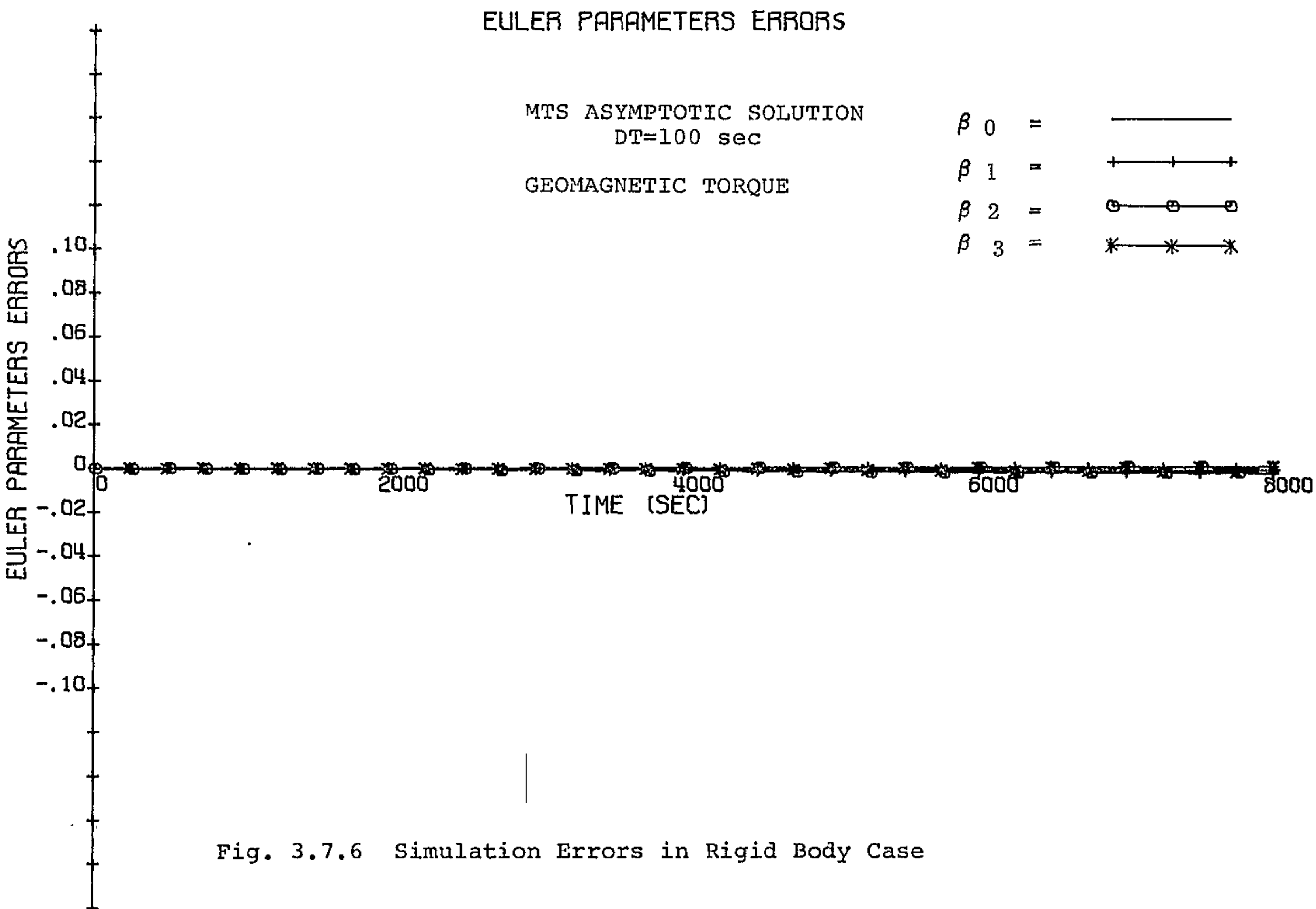


Fig. 3.7.6 Simulation Errors in Rigid Body Case

RUN NO.= 2686

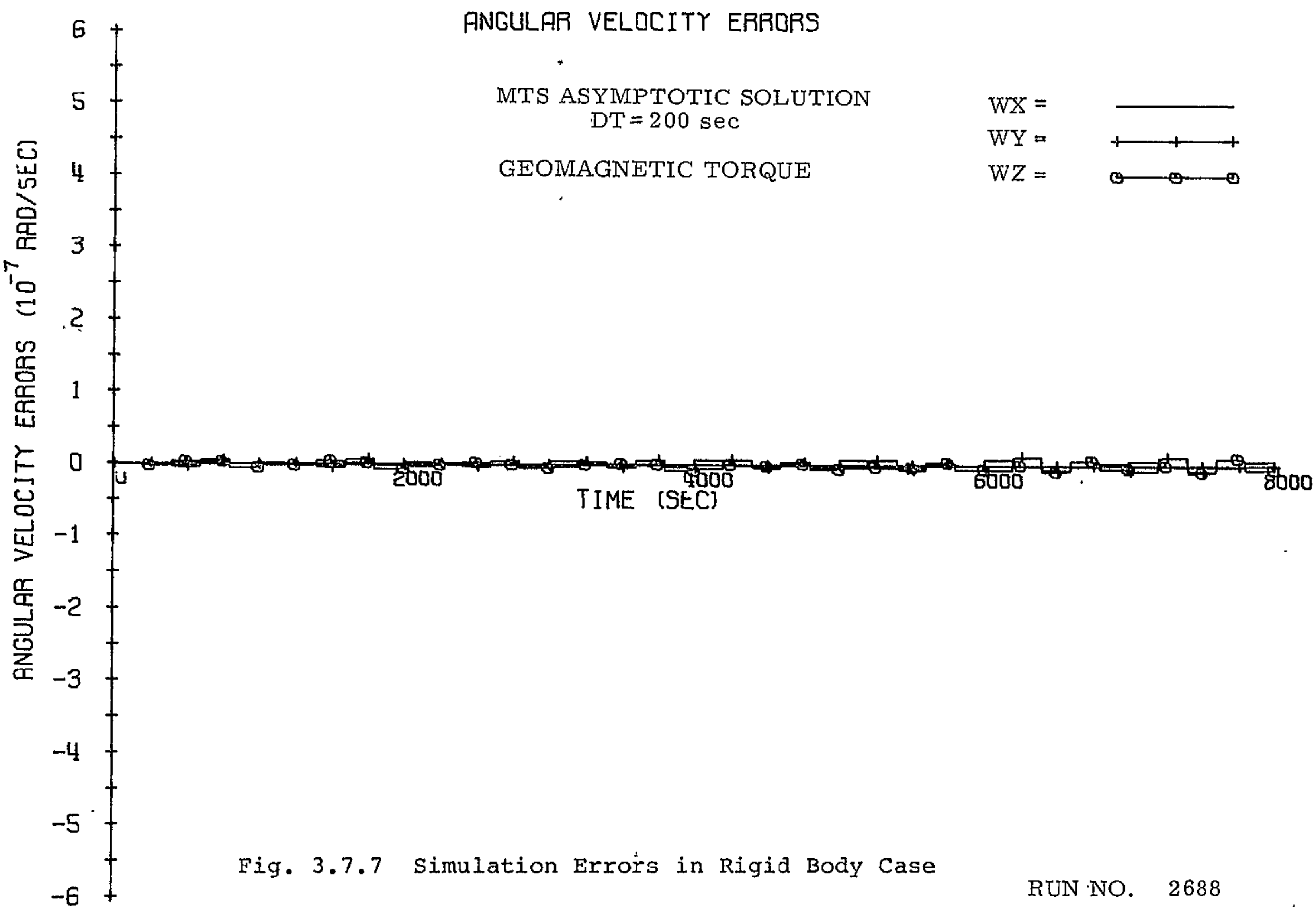


Fig. 3.7.7 Simulation Errors in Rigid Body Case

RUN NO. 2688

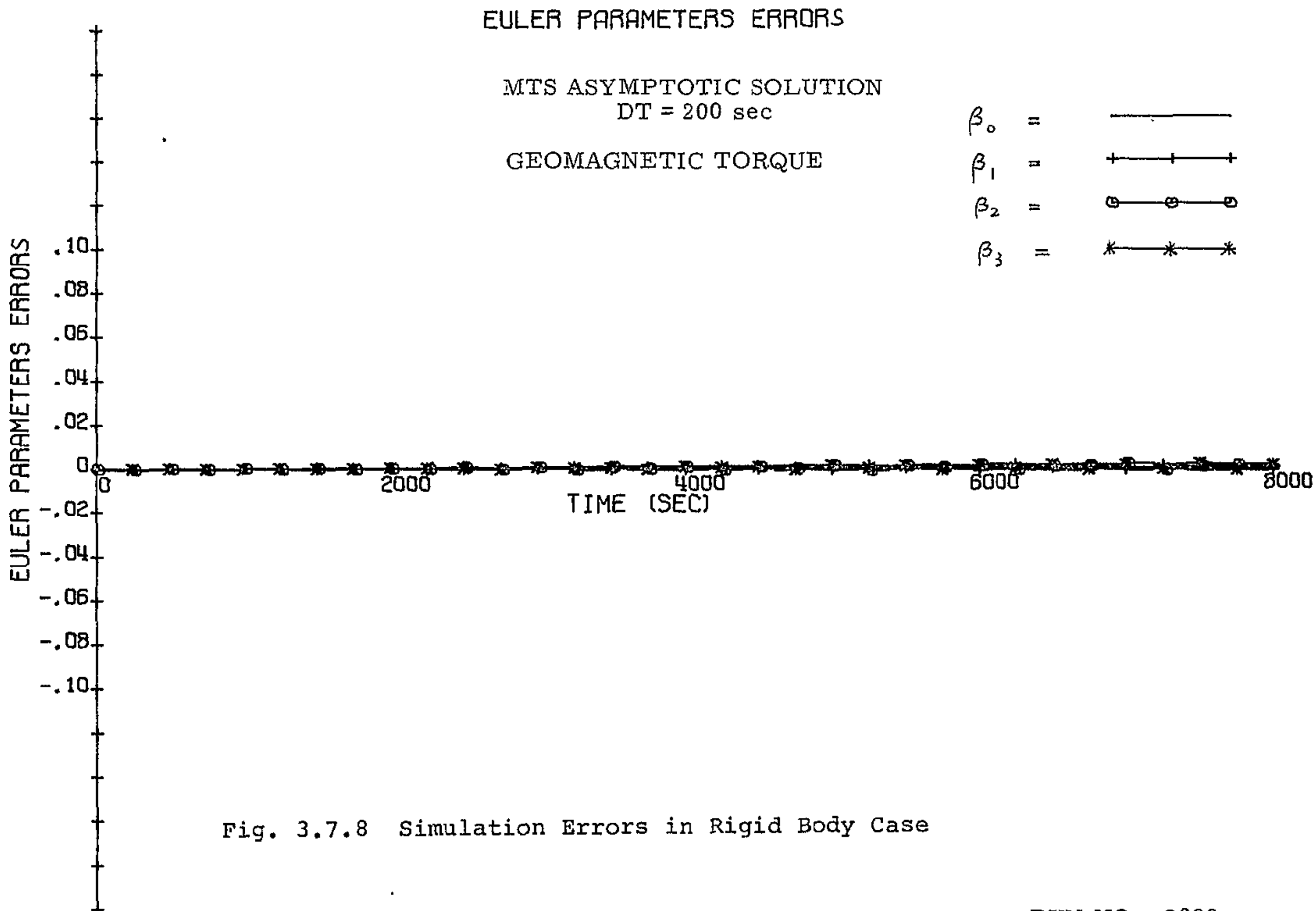


Fig. 3.7.8 Simulation Errors in Rigid Body Case

RUN NO. 2688

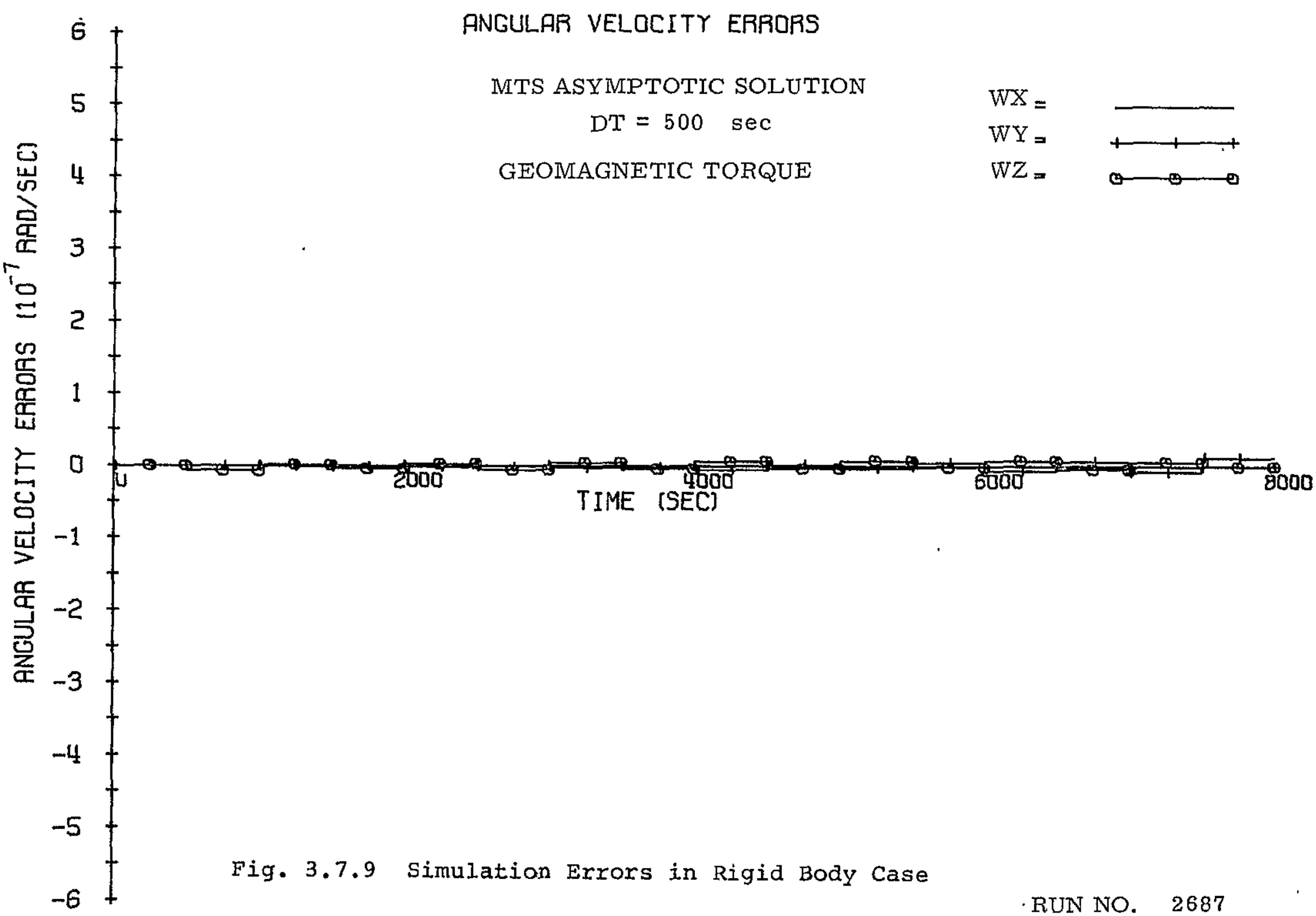


Fig. 3.7.9 Simulation Errors in Rigid Body Case

RUN NO. 2687

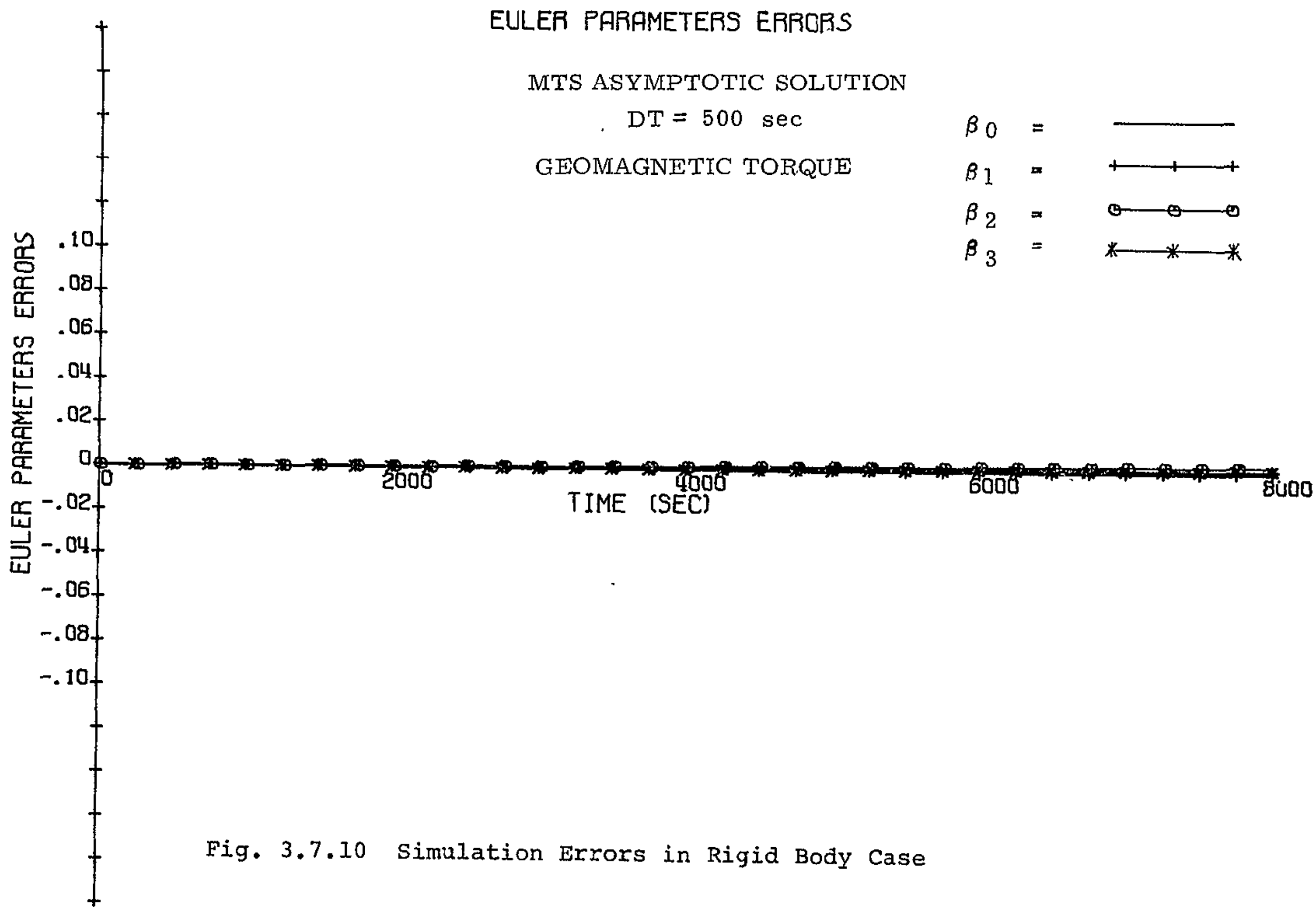


Fig. 3.7.10 Simulation Errors in Rigid Body Case

RUN NO. 2687

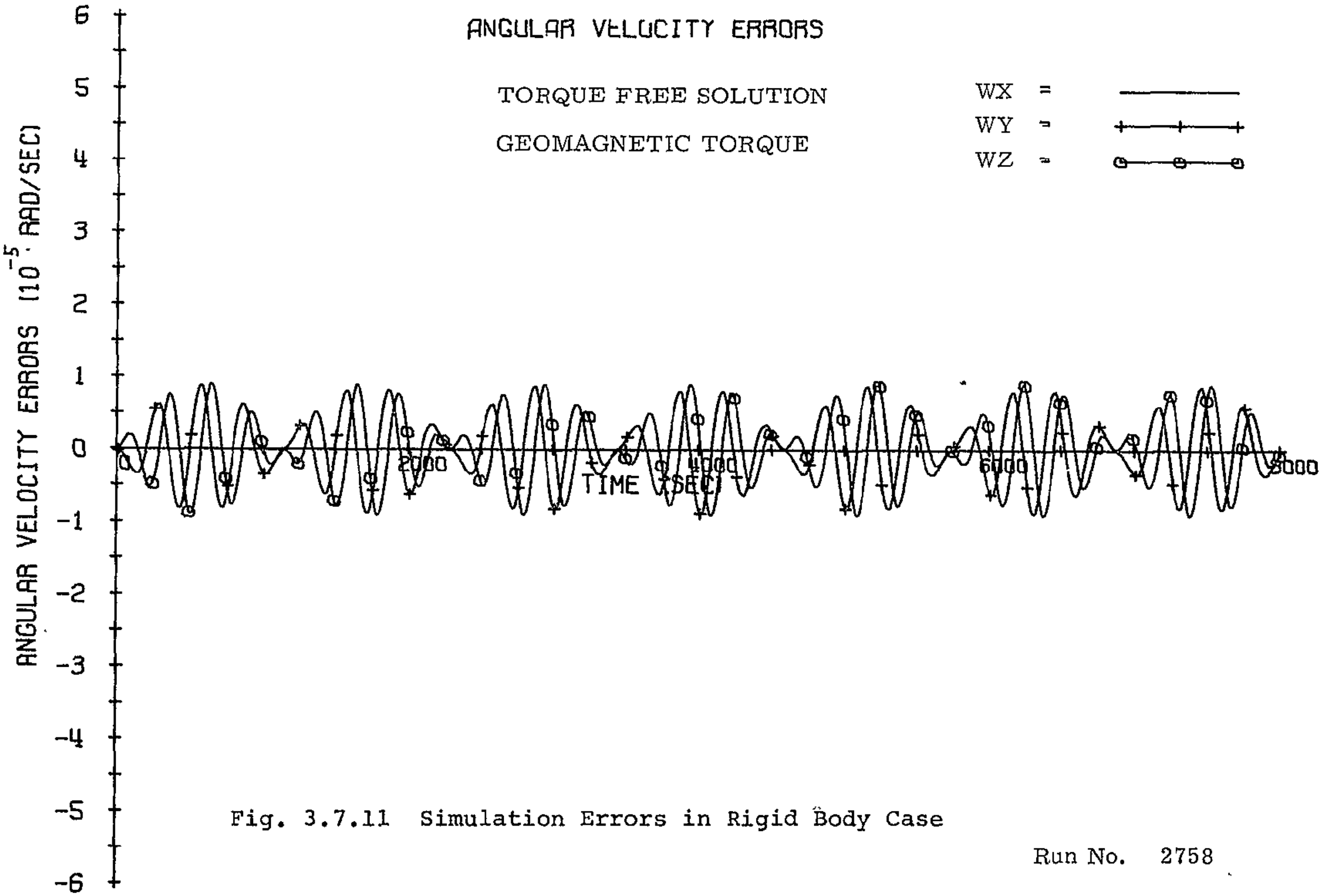


Fig. 3.7.11 Simulation Errors in Rigid Body Case

Run No. 2758

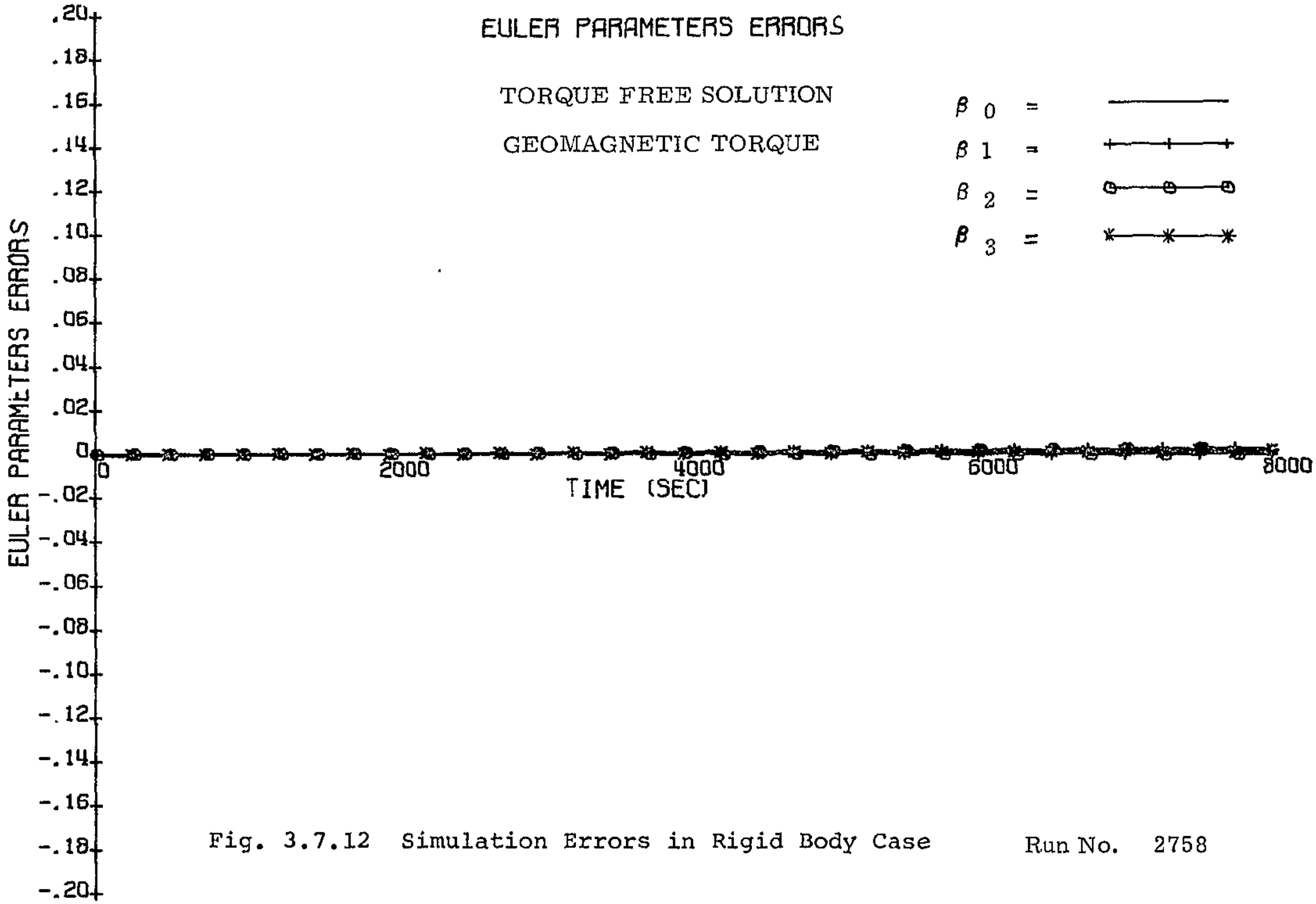


Fig. 3.7.12 Simulation Errors in Rigid Body Case

Run No. 2758

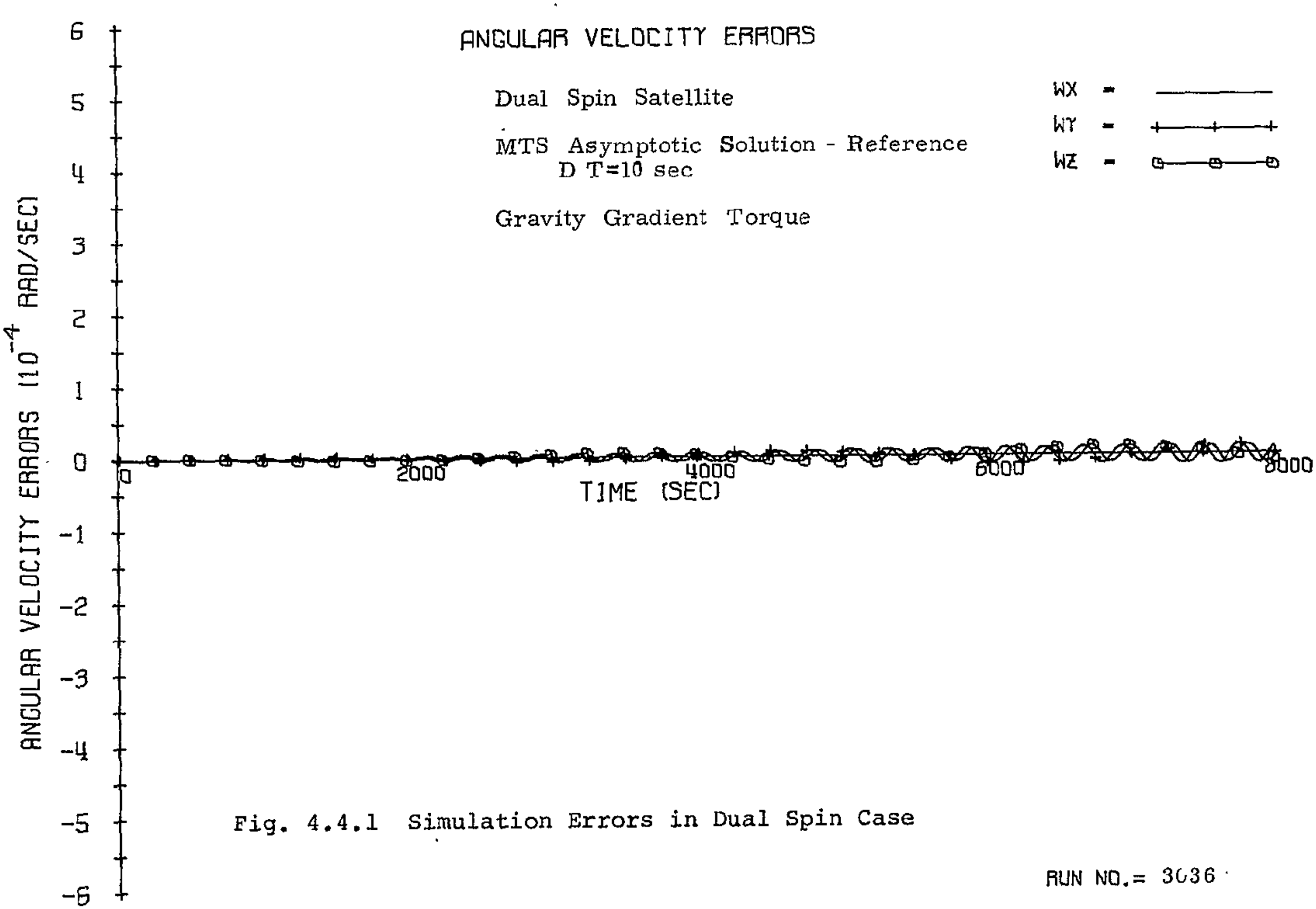


Fig. 4.4.1 Simulation Errors in Dual Spin Case

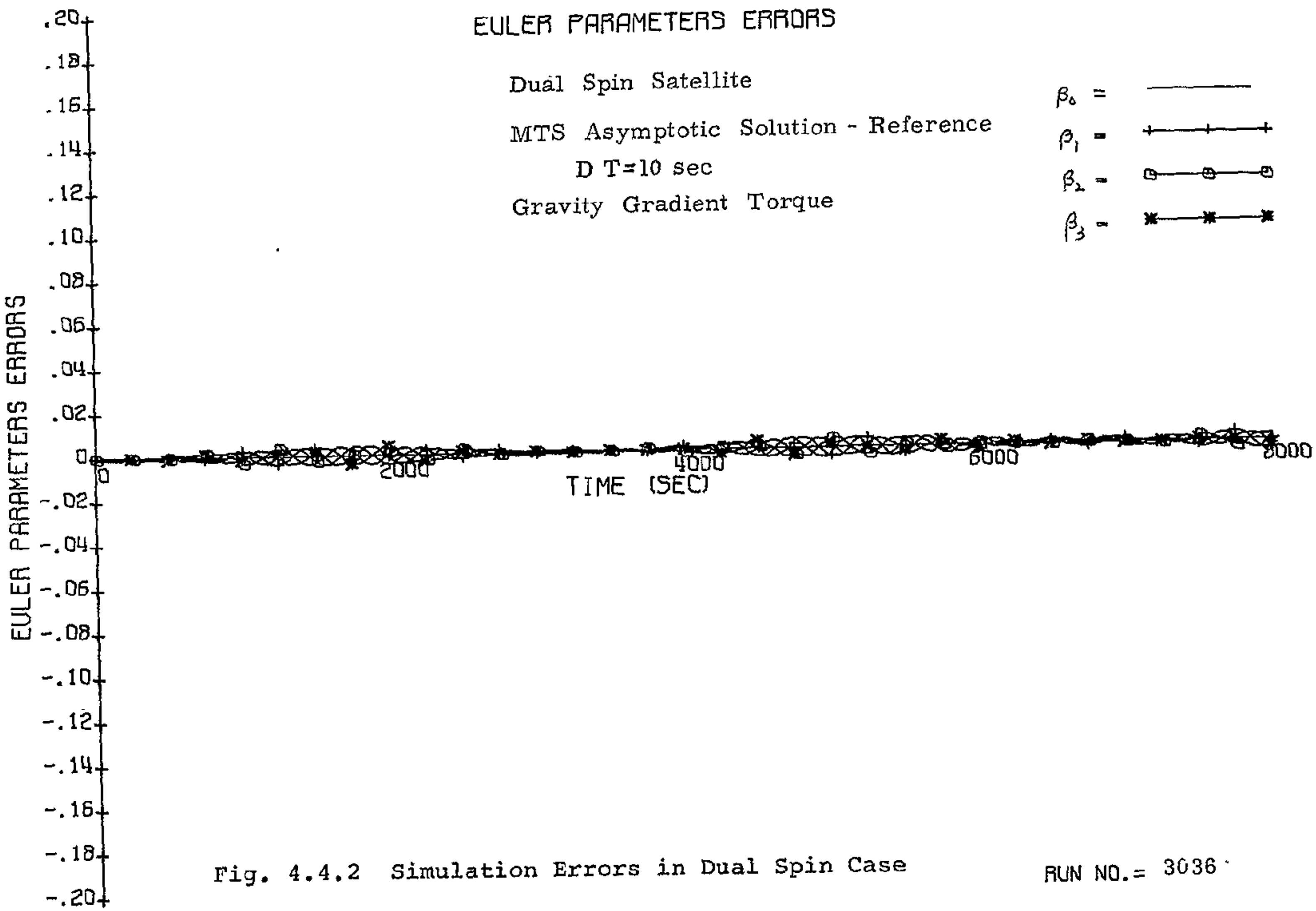


Fig. 4.4.2 Simulation Errors in Dual Spin Case

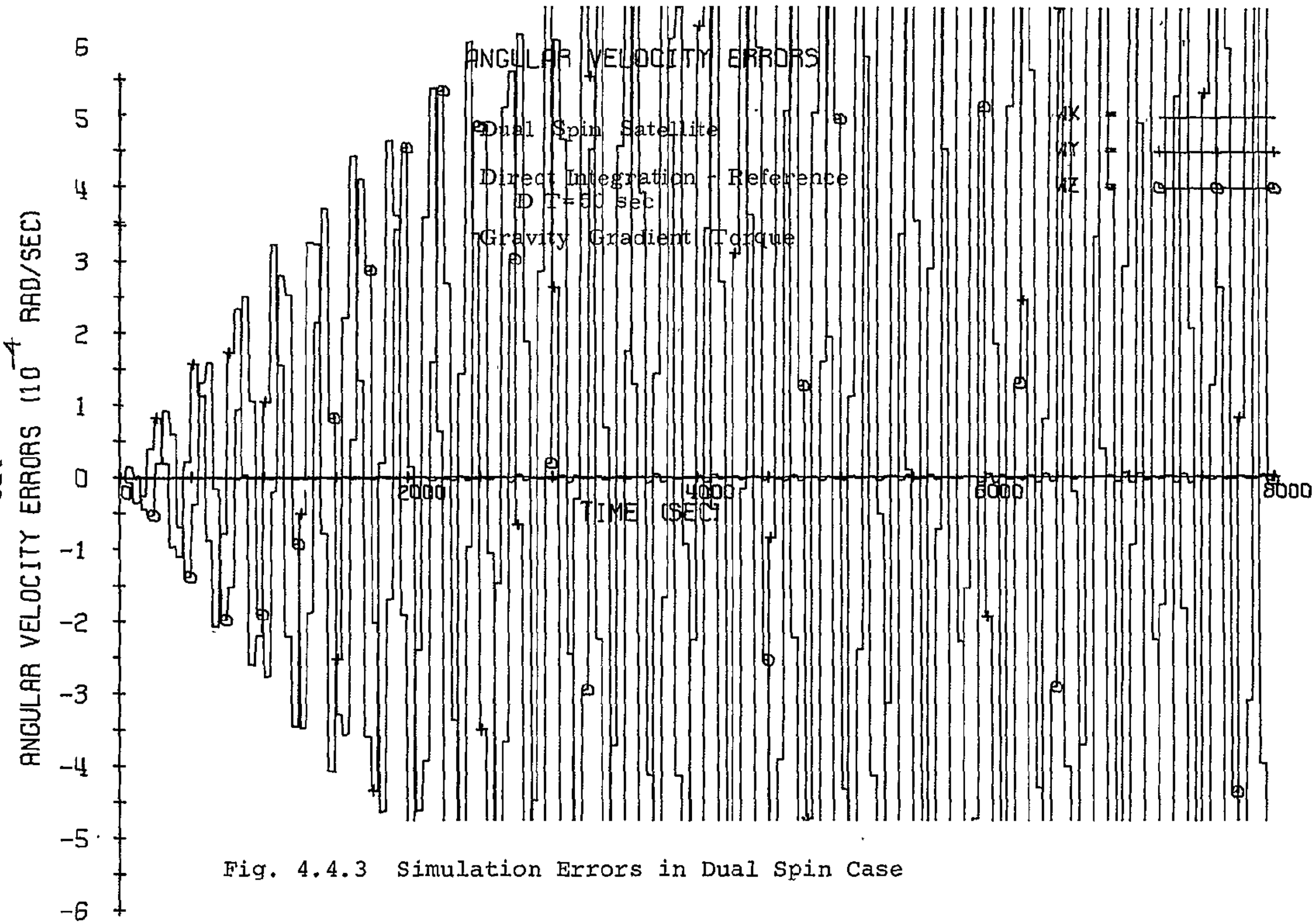


Fig. 4.4.3 Simulation Errors in Dual Spin Case

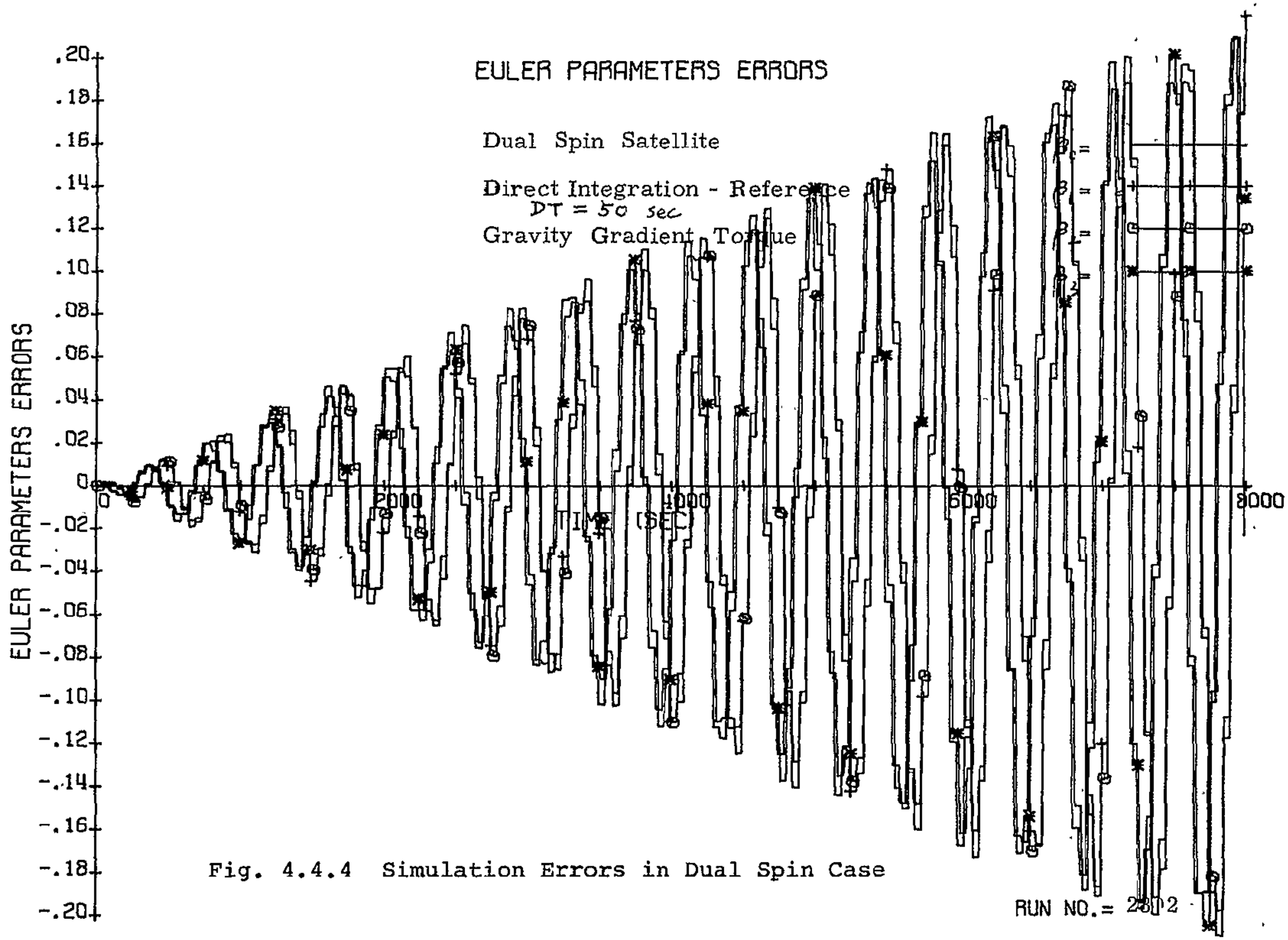


Fig. 4.4.4 Simulation Errors in Dual Spin Case

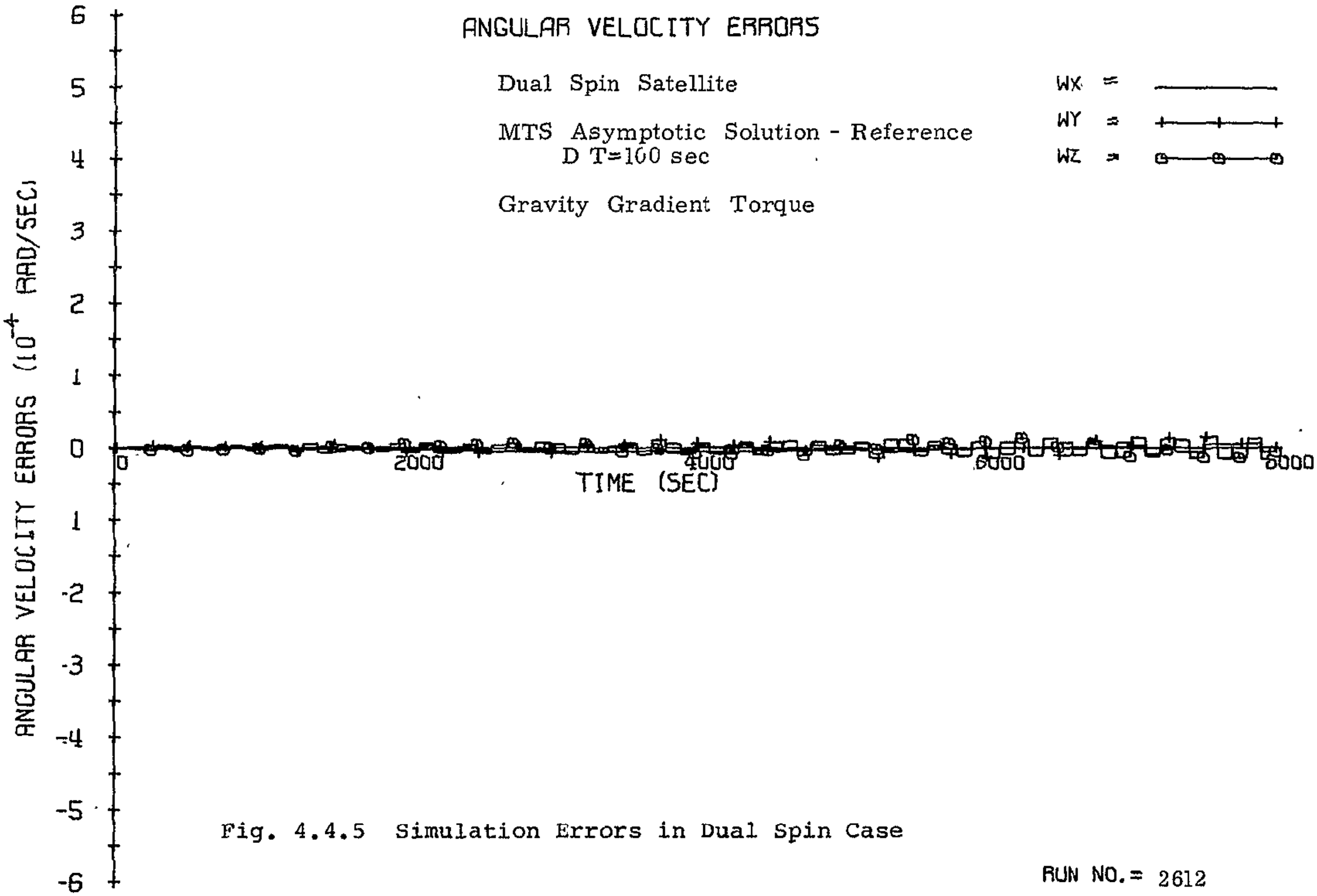


Fig. 4.4.5 Simulation Errors in Dual Spin Case

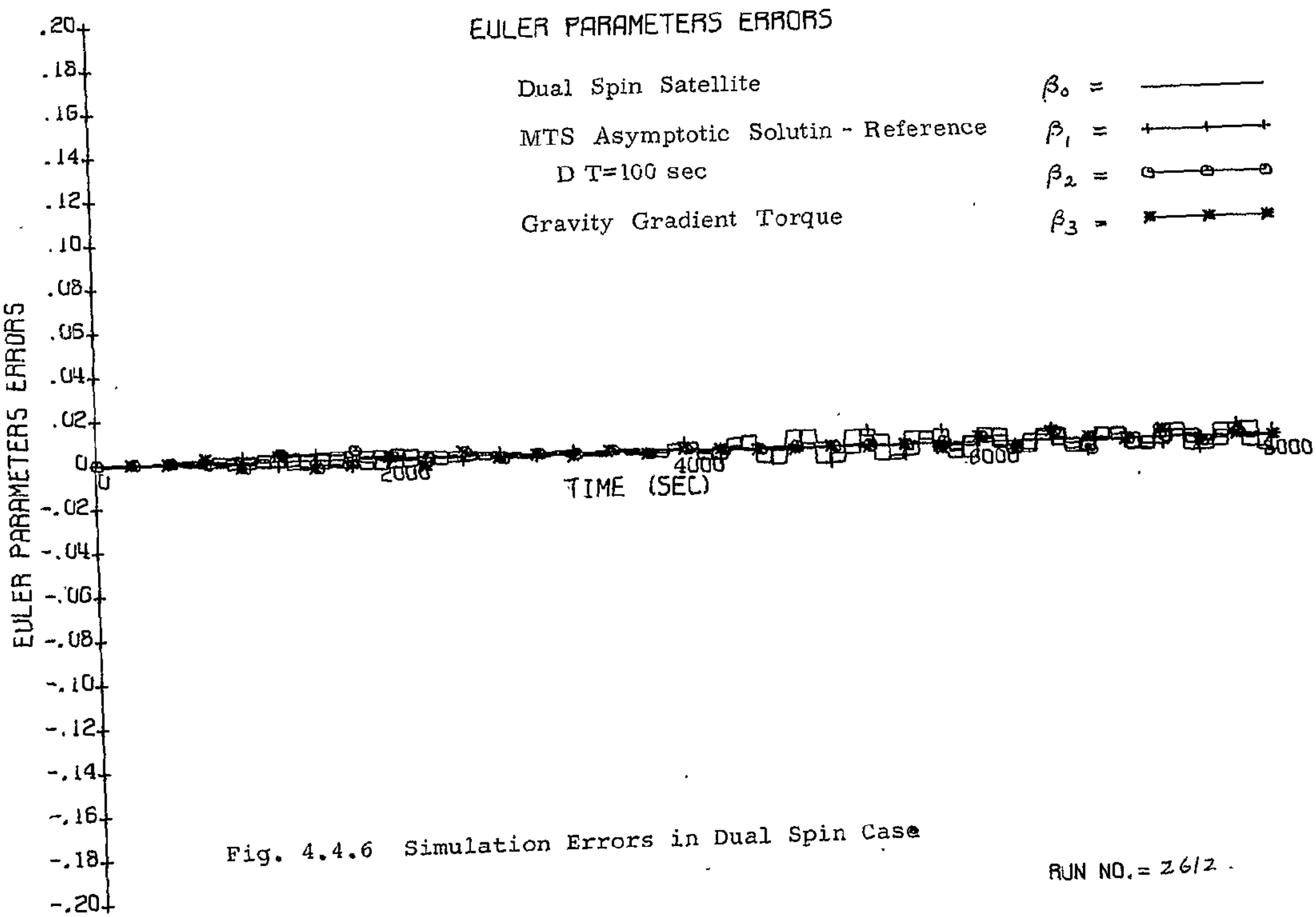


Fig. 4.4.6 Simulation Errors in Dual Spin Case

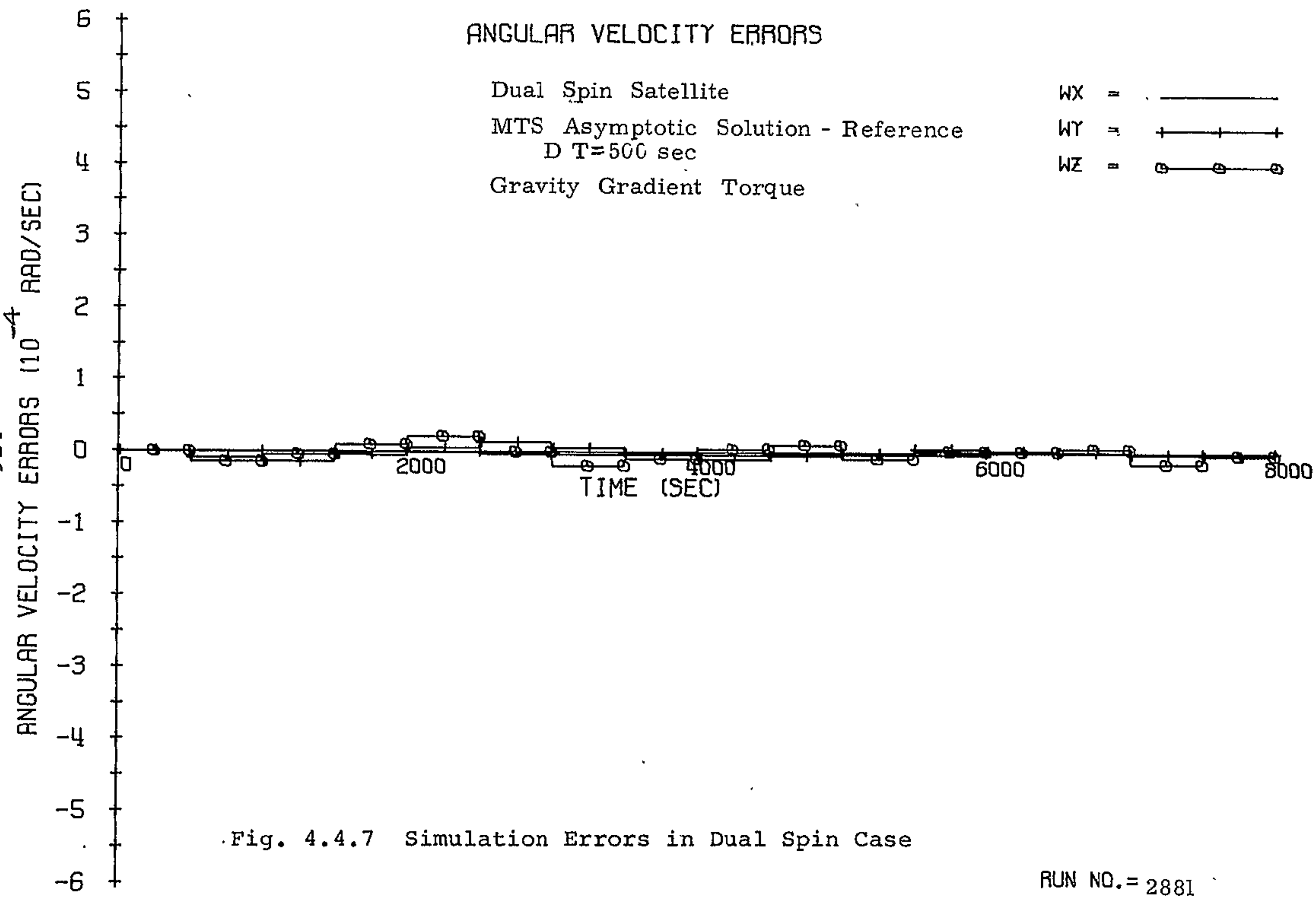


Fig. 4.4.7 Simulation Errors in Dual Spin Case

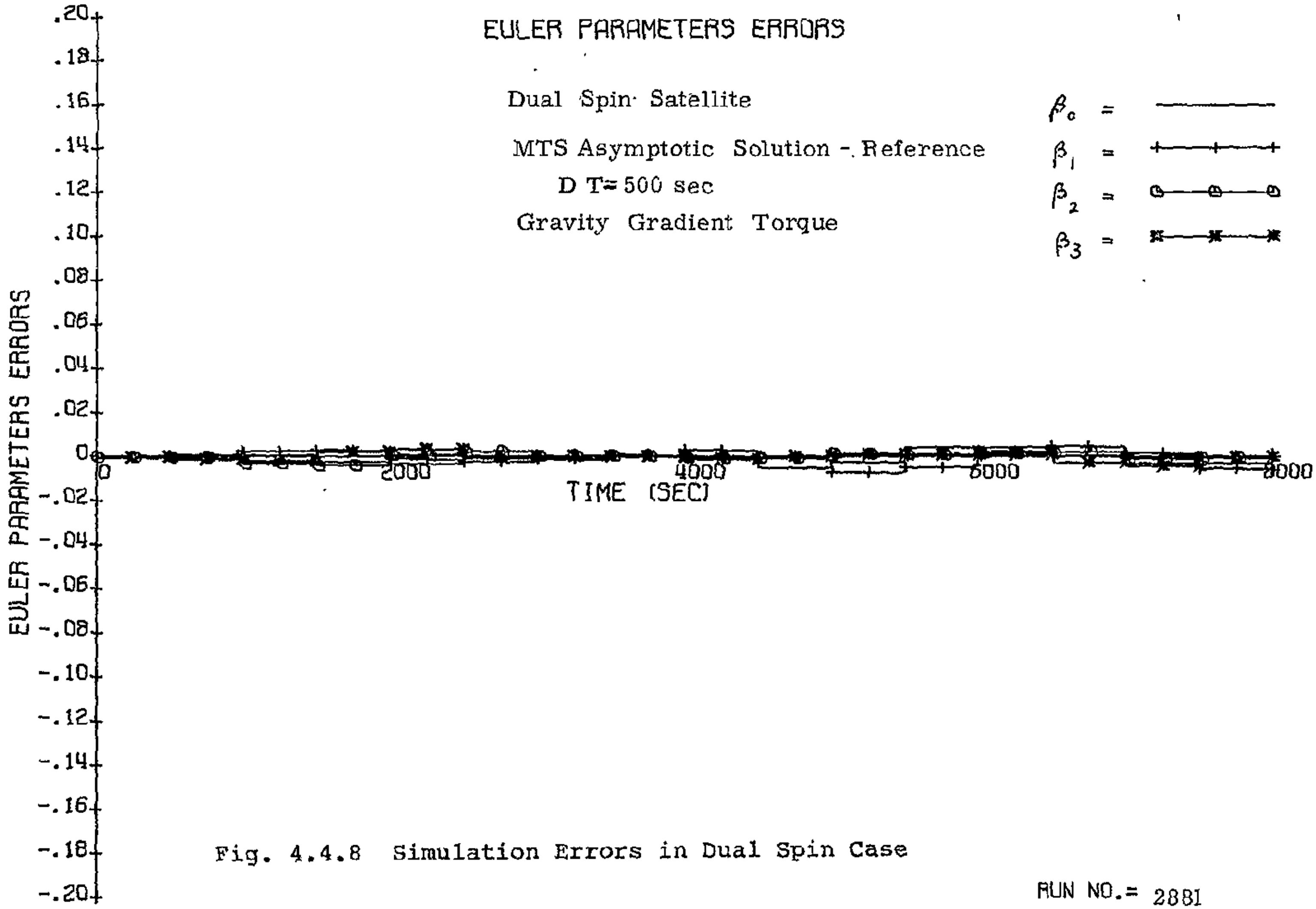


Fig. 4.4.8 Simulation Errors in Dual Spin Case

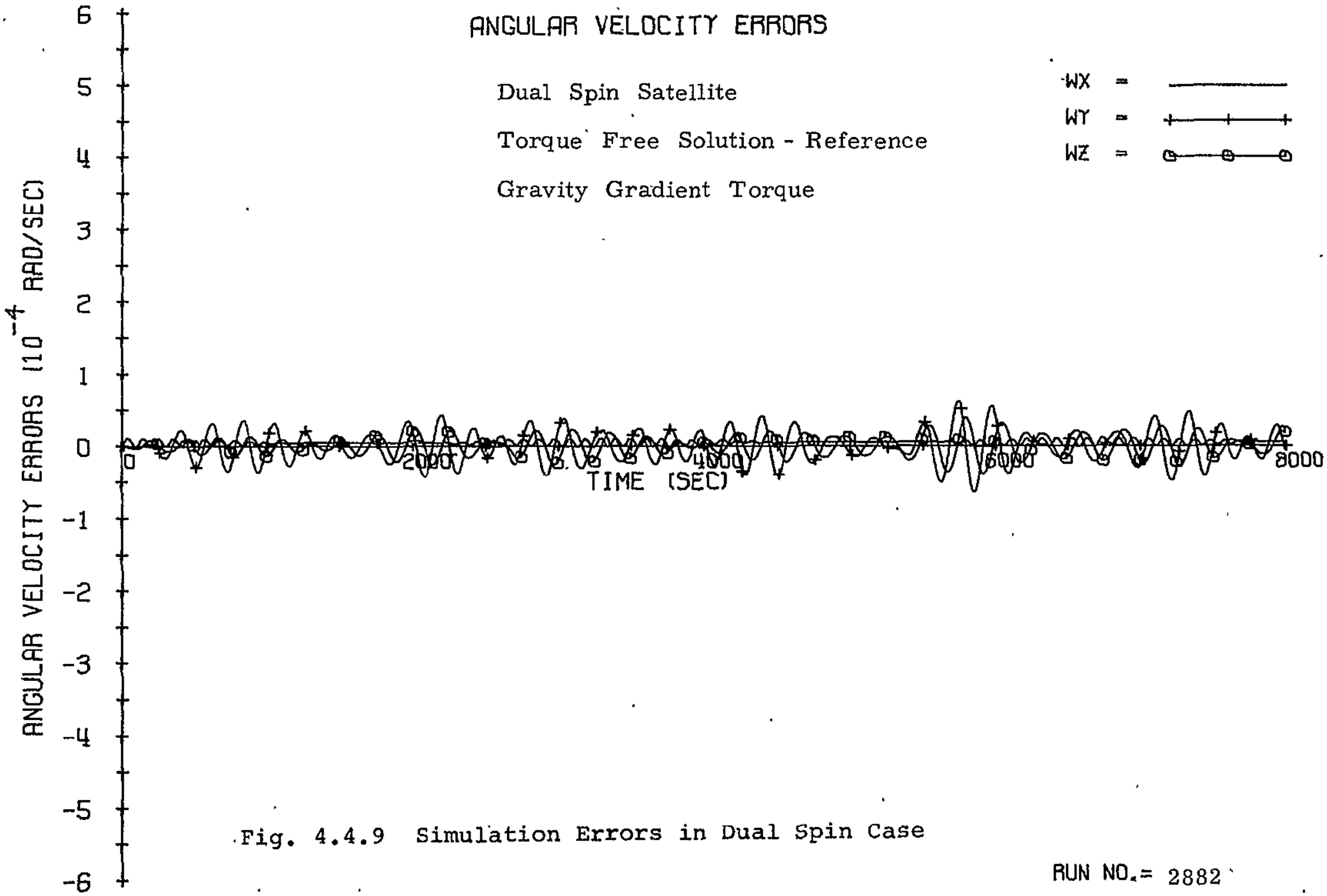


Fig. 4.4.9 Simulation Errors in Dual Spin Case

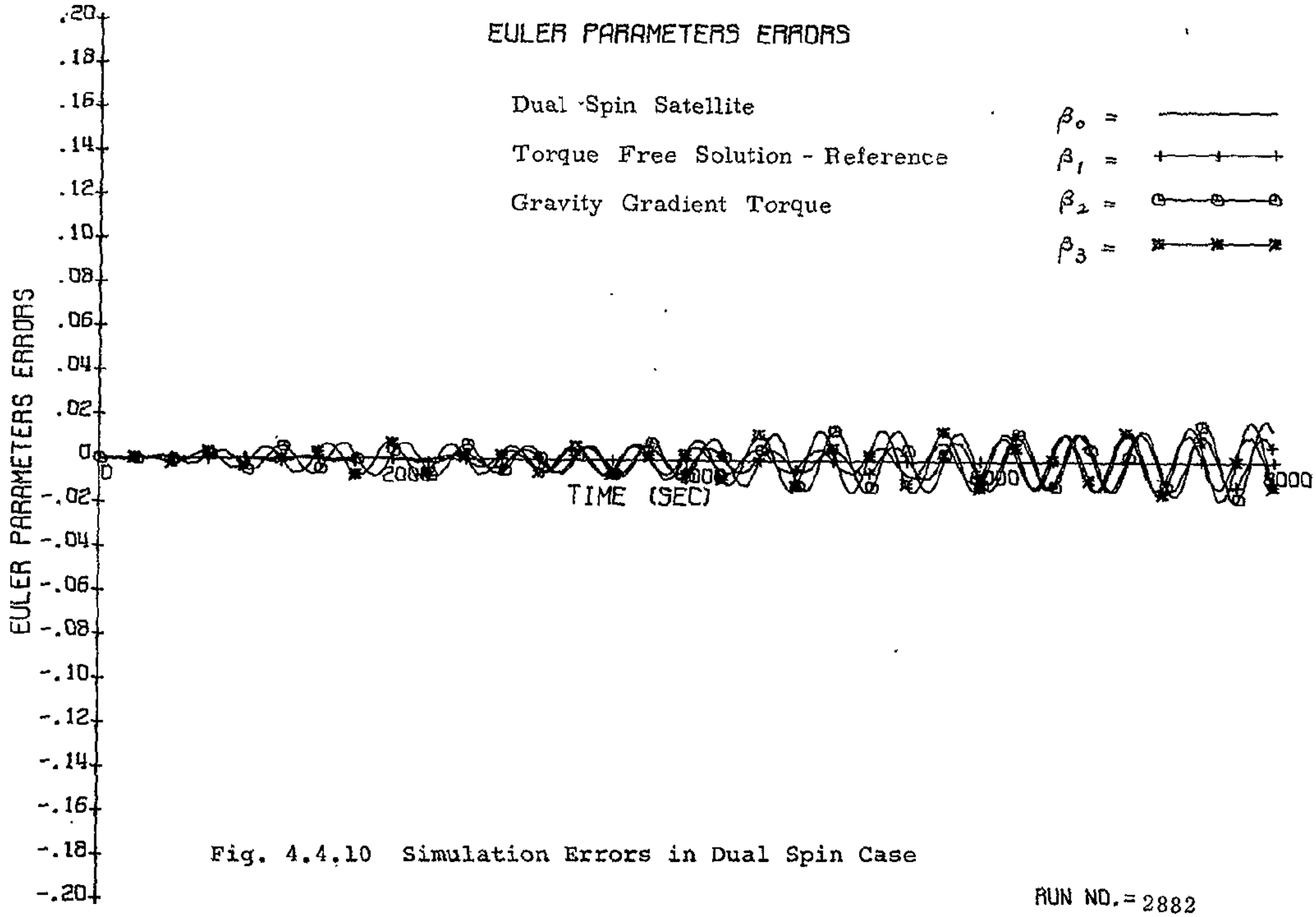


Fig. 4.4.10 Simulation Errors in Dual Spin Case

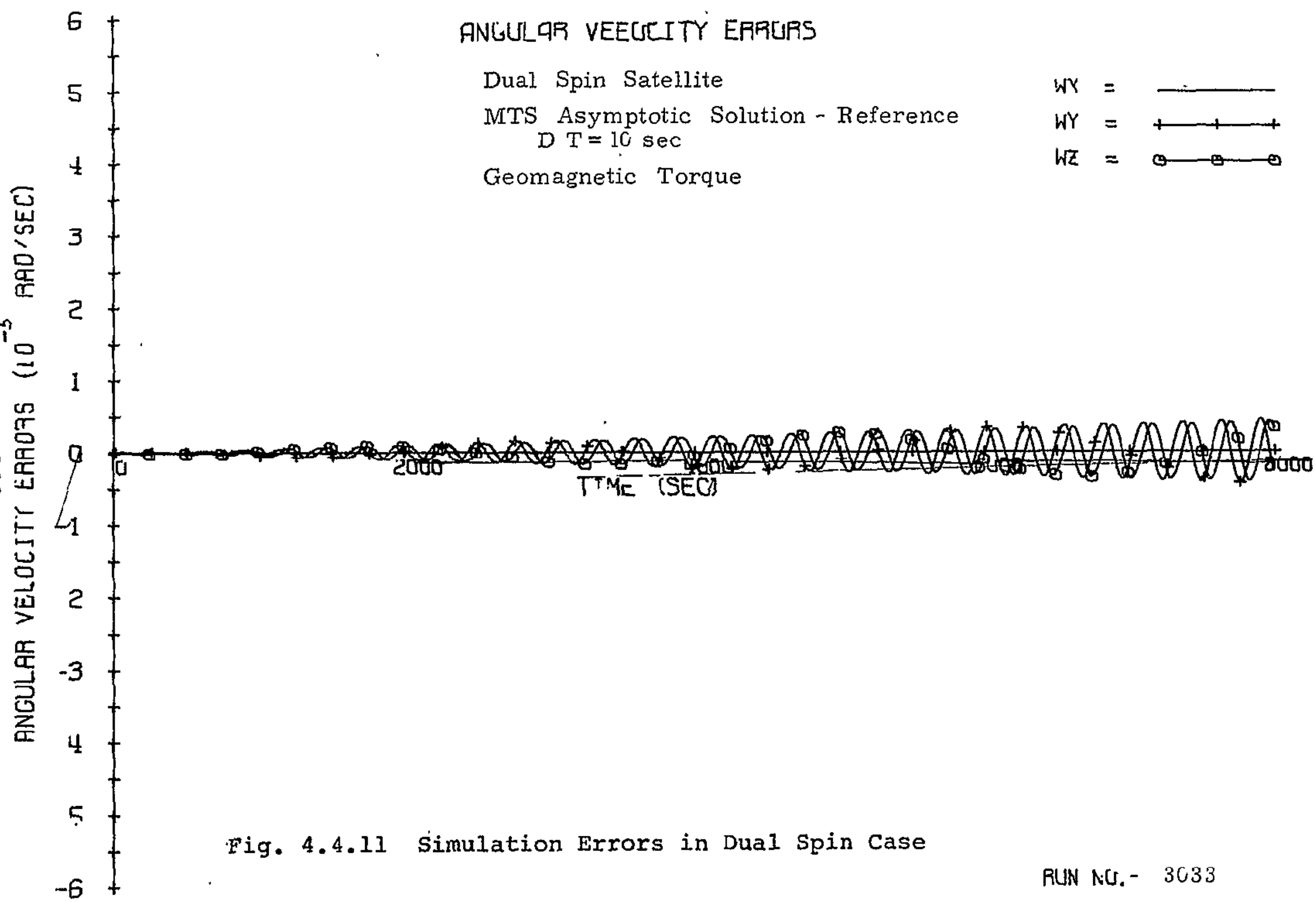


Fig. 4.4.11 Simulation Errors in Dual Spin Case

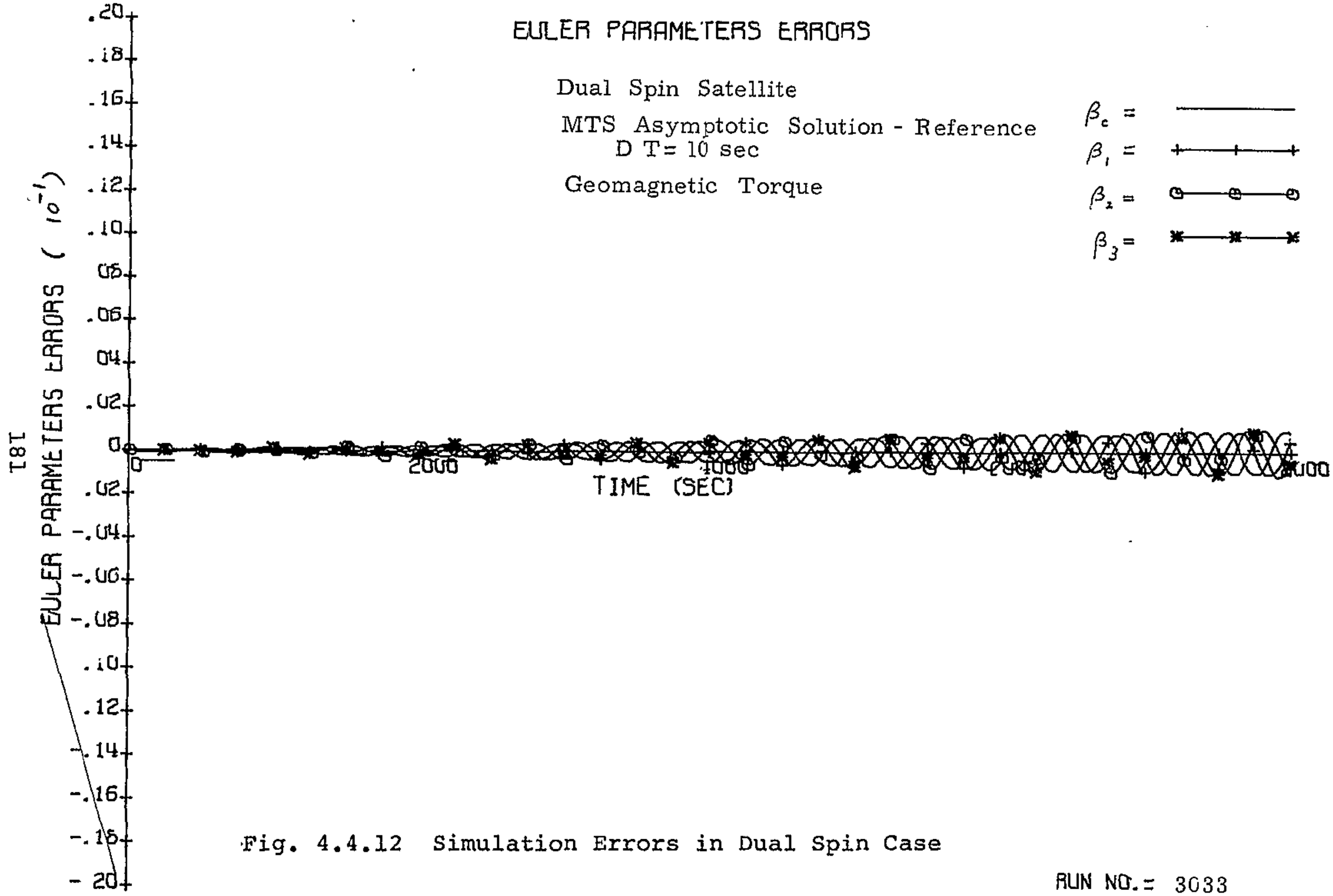


Fig. 4.4.12 Simulation Errors in Dual Spin Case

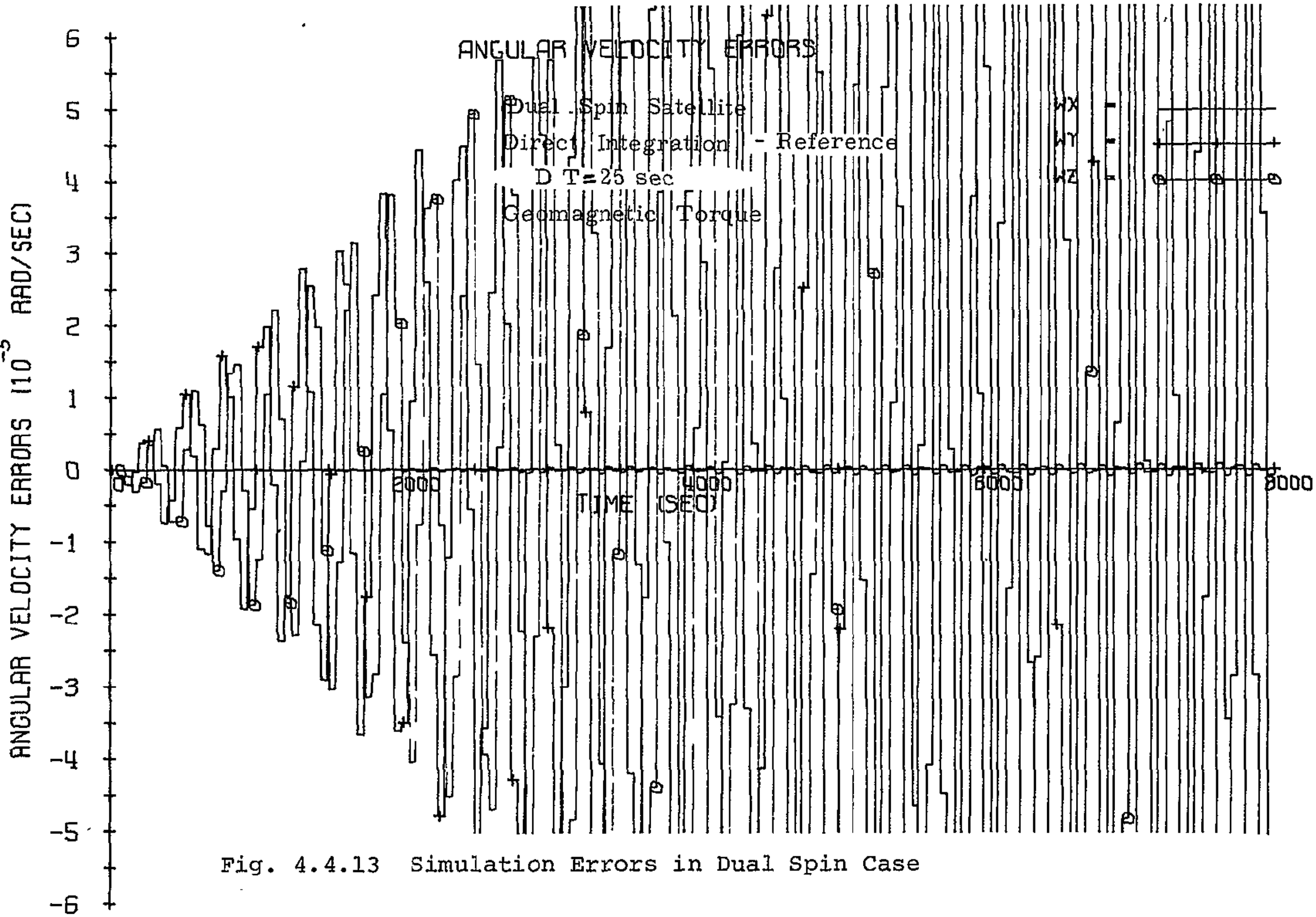


Fig. 4.4.13 Simulation Errors in Dual Spin Case

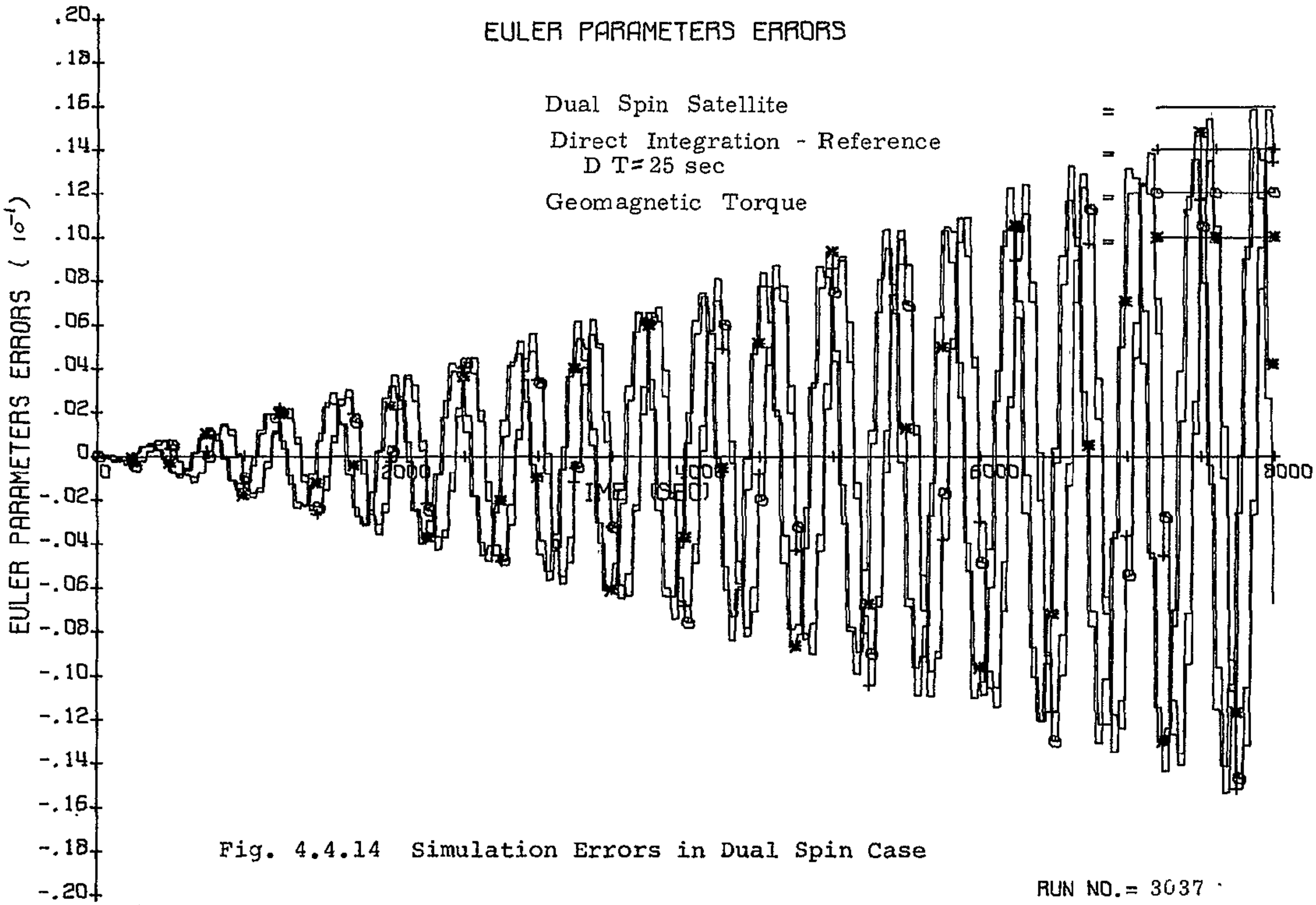


Fig. 4.4.14 Simulation Errors in Dual Spin Case

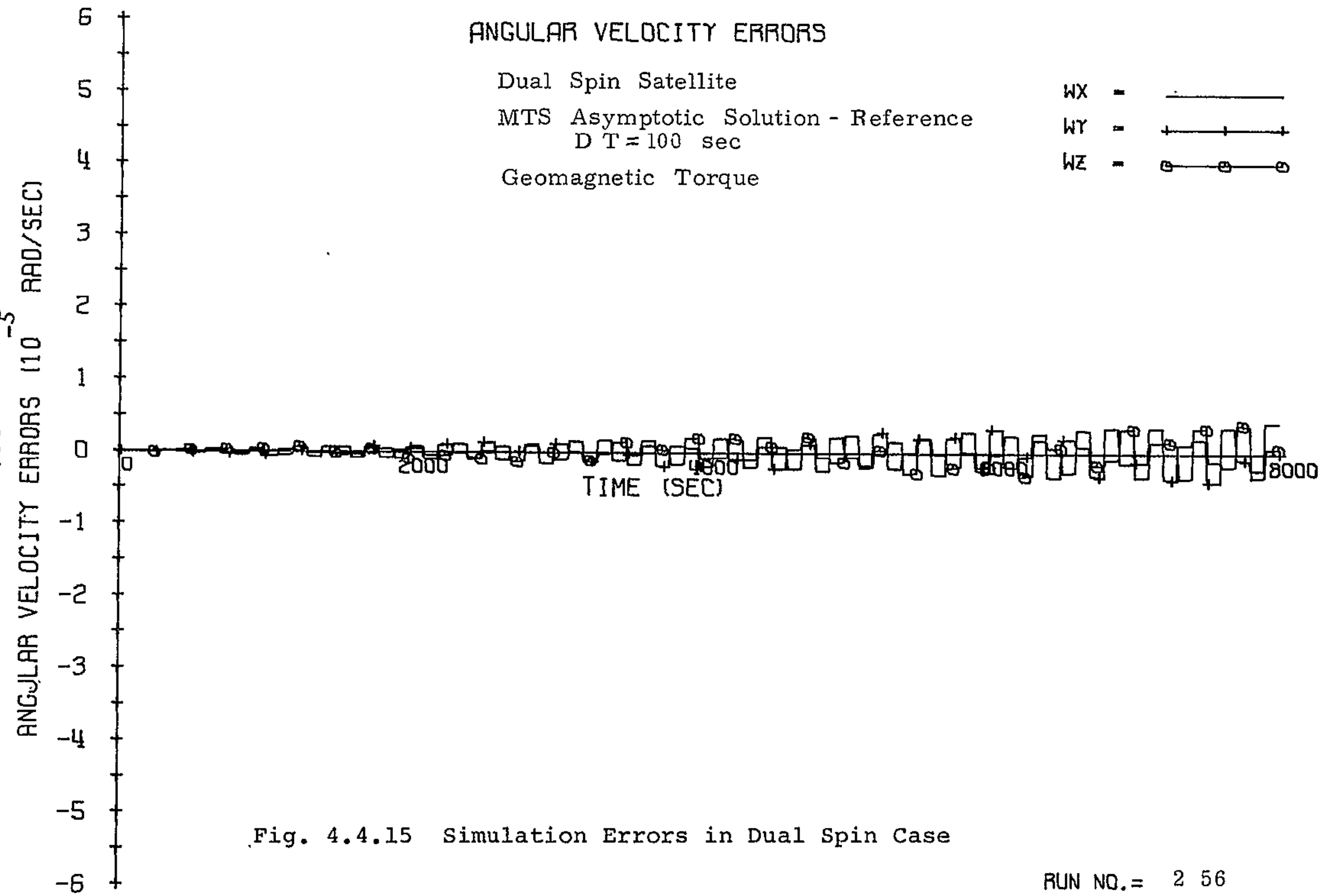


Fig. 4.4.15 Simulation Errors in Dual Spin Case

RUN NO.= 2 56

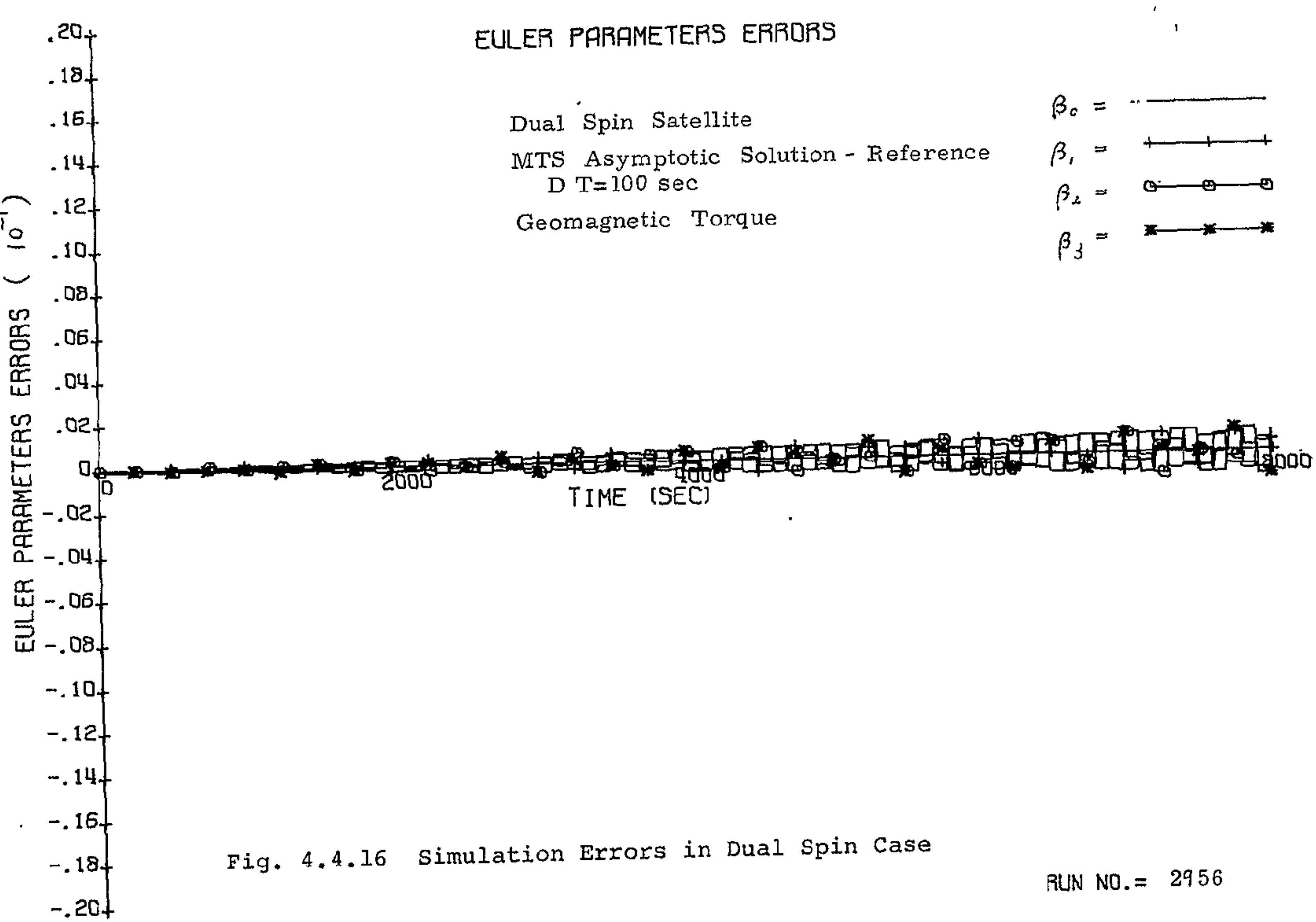


Fig. 4.4.16 Simulation Errors in Dual Spin Case

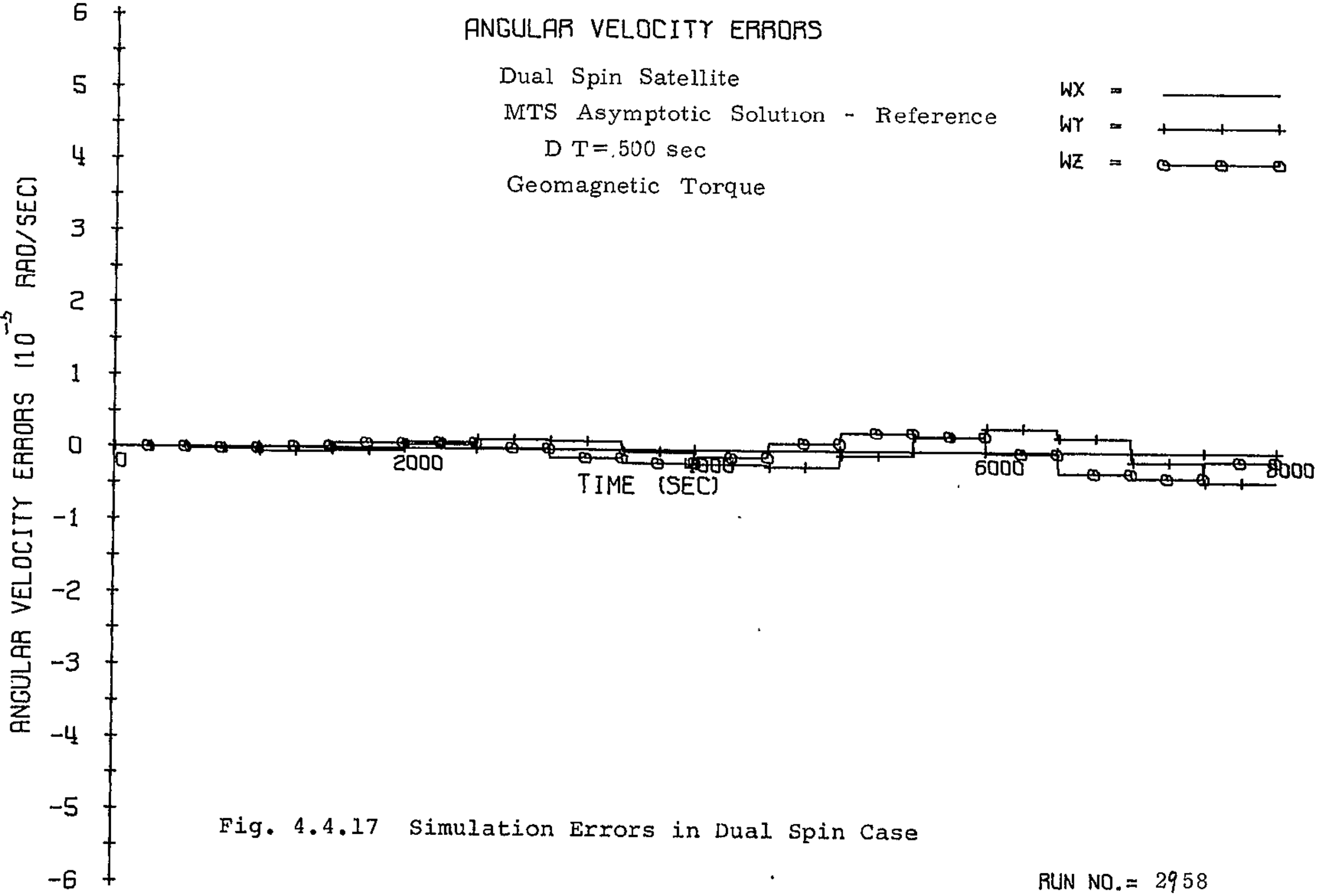


Fig. 4.4.17 Simulation Errors in Dual Spin Case

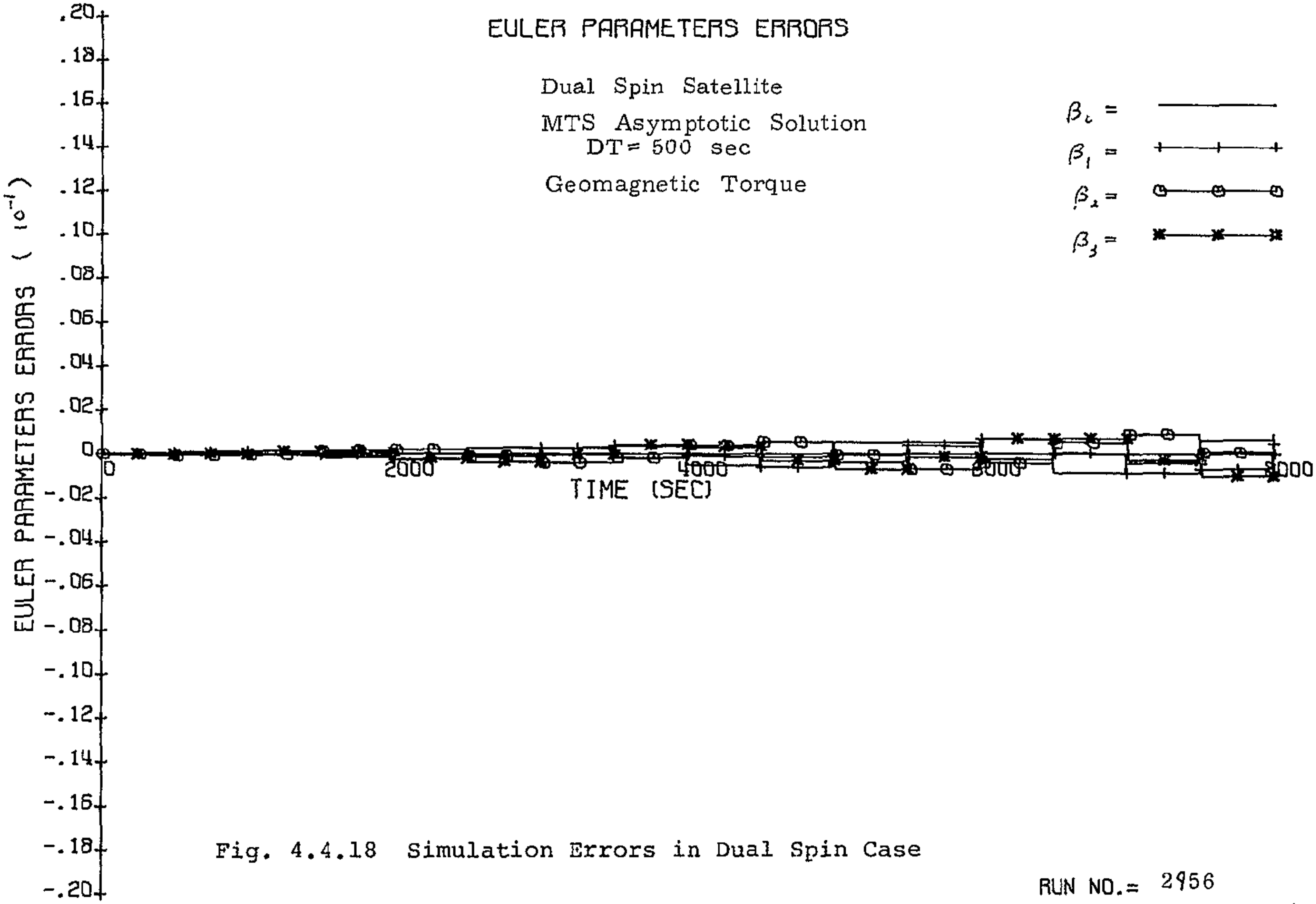


Fig. 4.4.18 Simulation Errors in Dual Spin Case

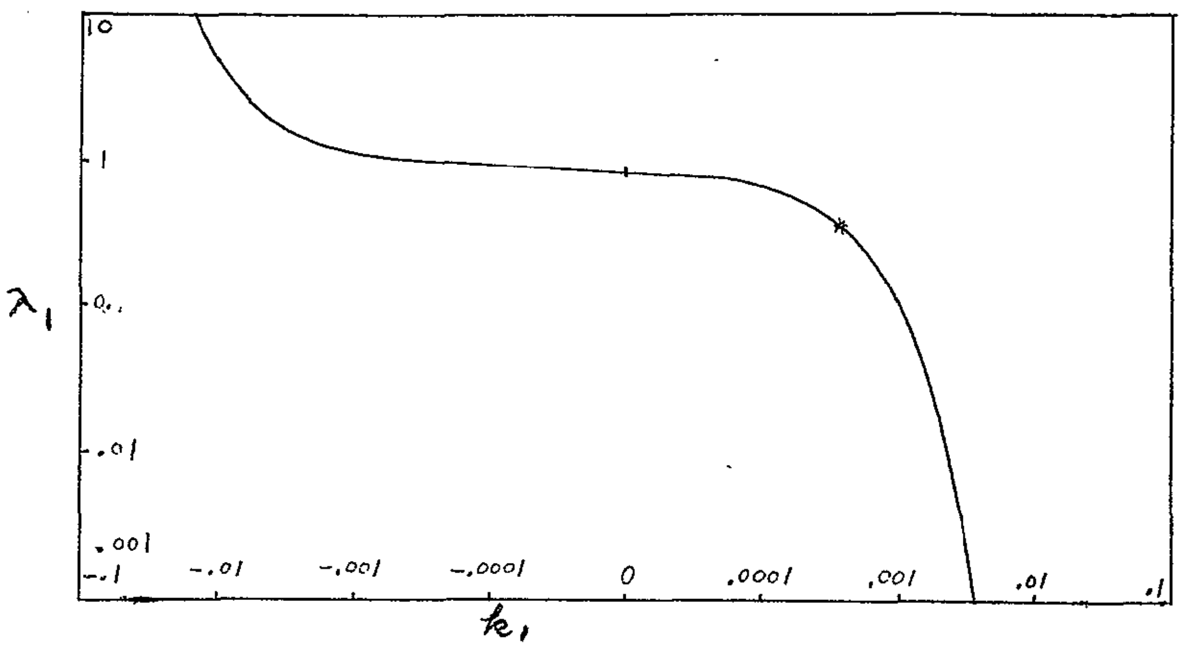


Fig. 5.4.1 Eigenvalue λ_1

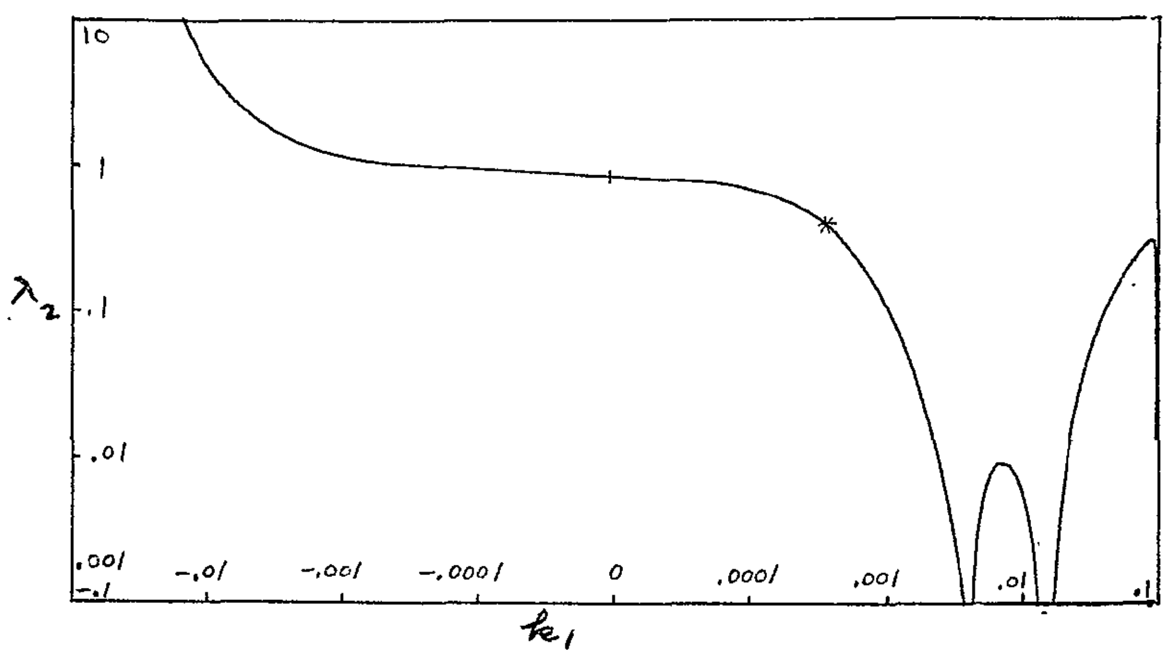


Fig. 5.4.2 Eigenvalue λ_2

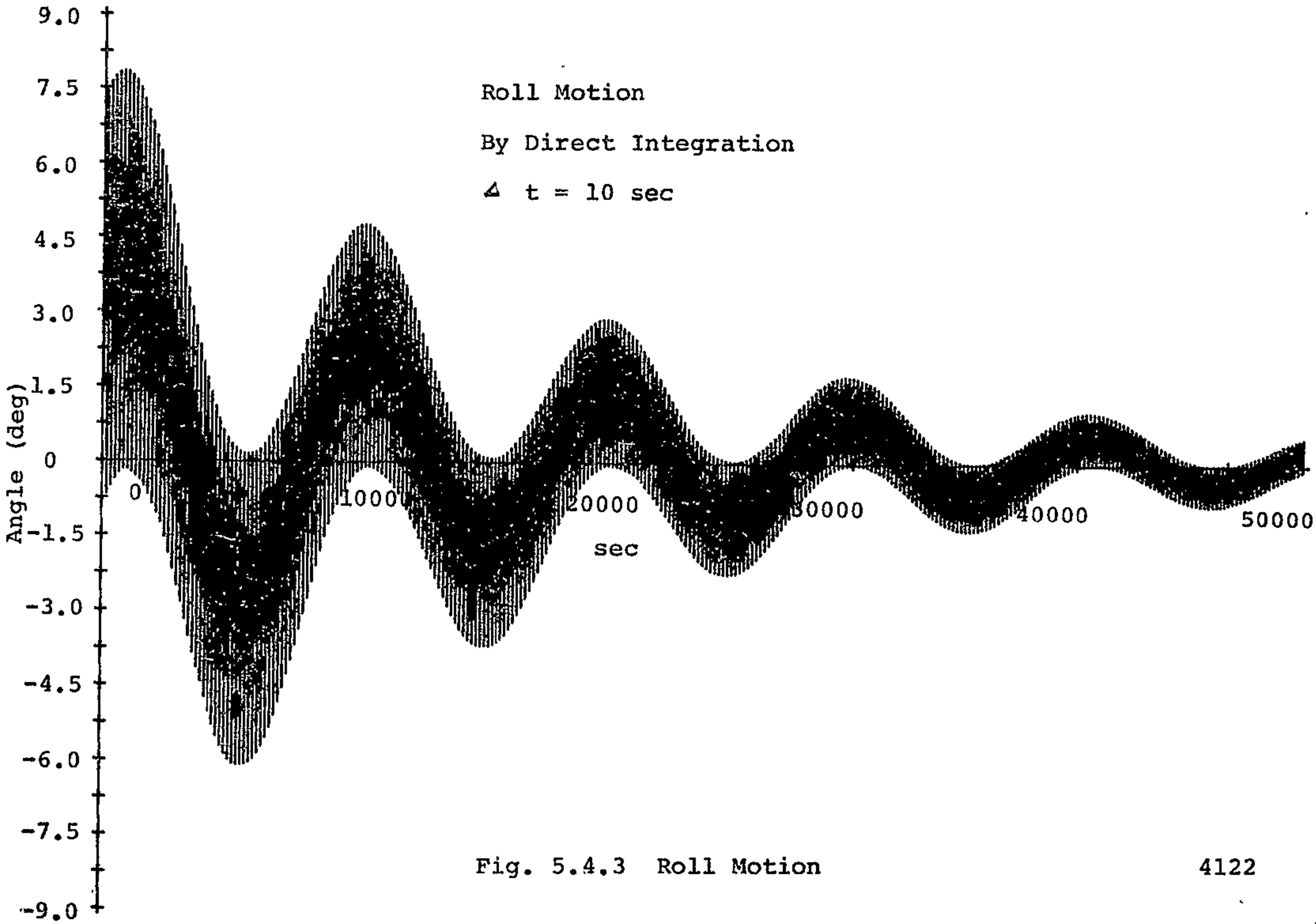


Fig. 5.4.3 Roll Motion 4122

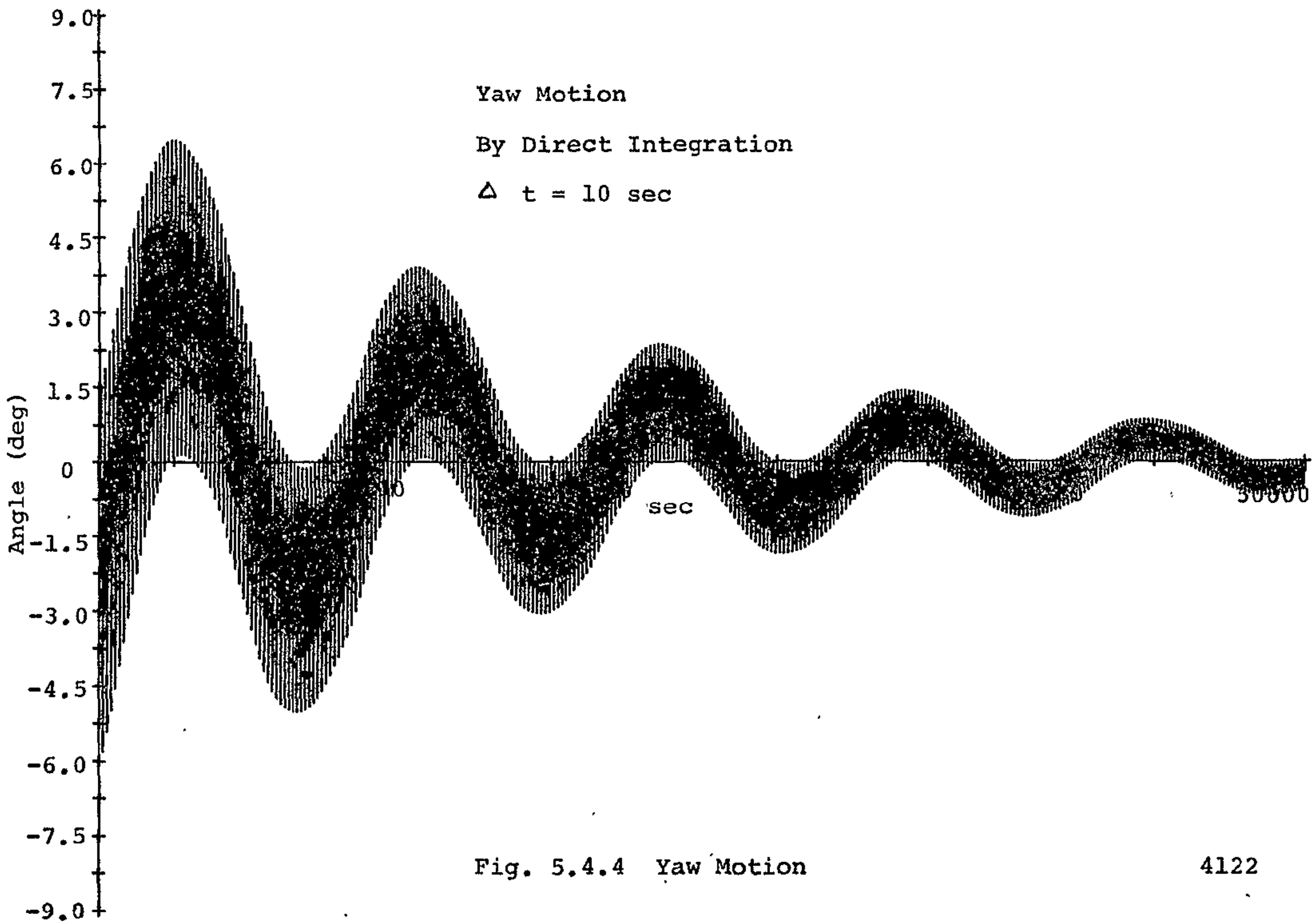


Fig. 5.4.4 Yaw Motion 4122

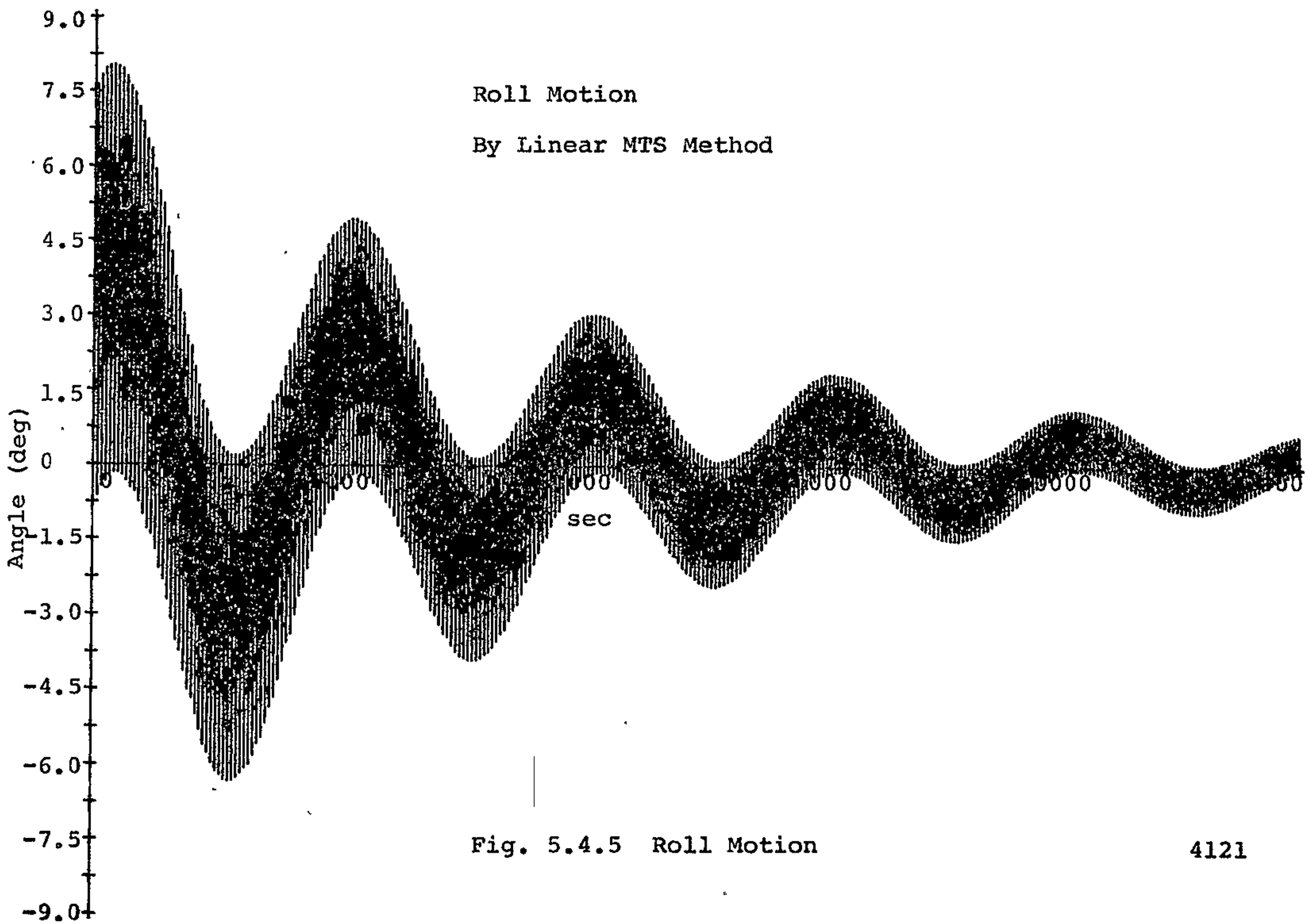


Fig. 5.4.5 Roll Motion

4121

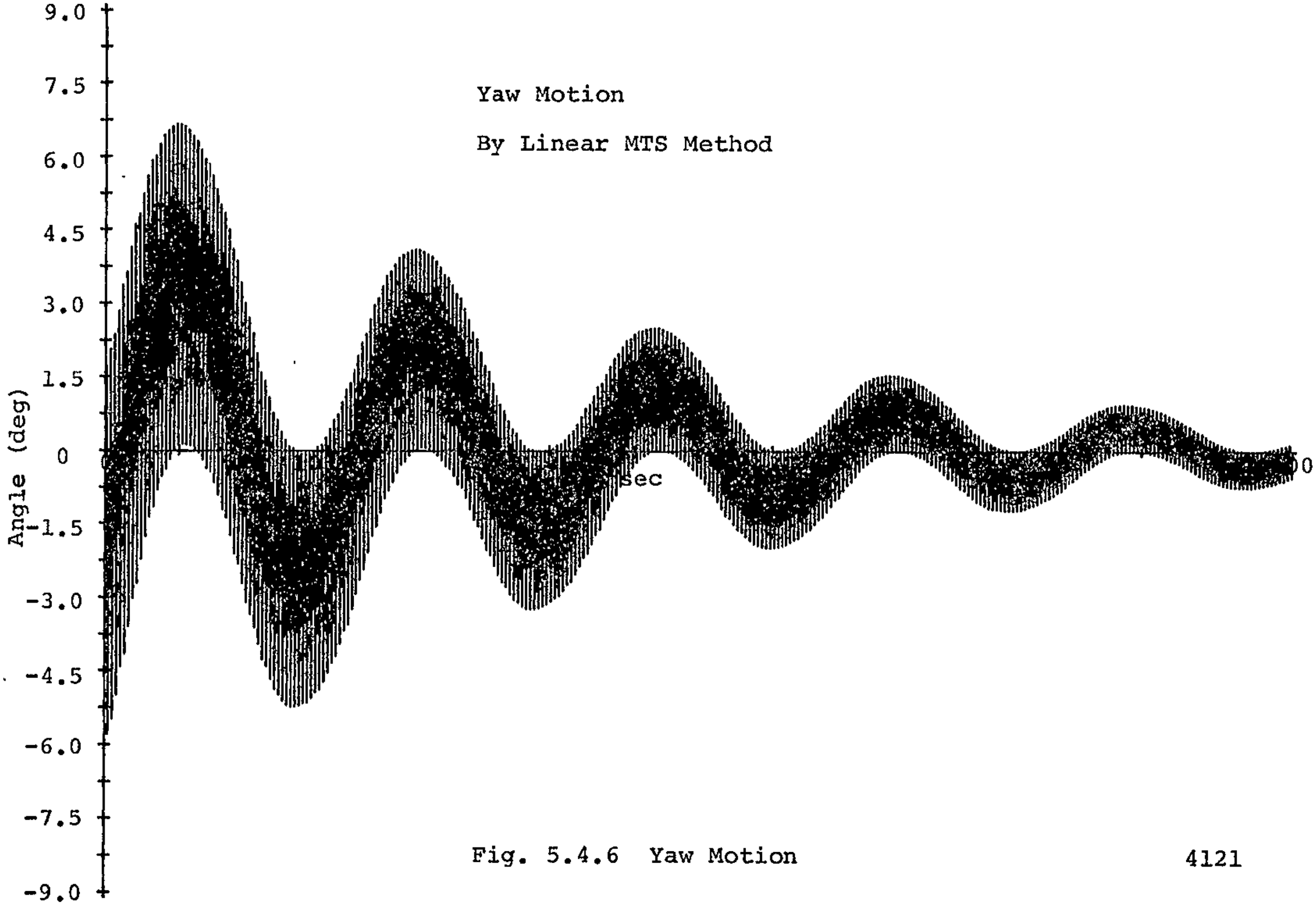


Fig. 5.4.6 Yaw Motion

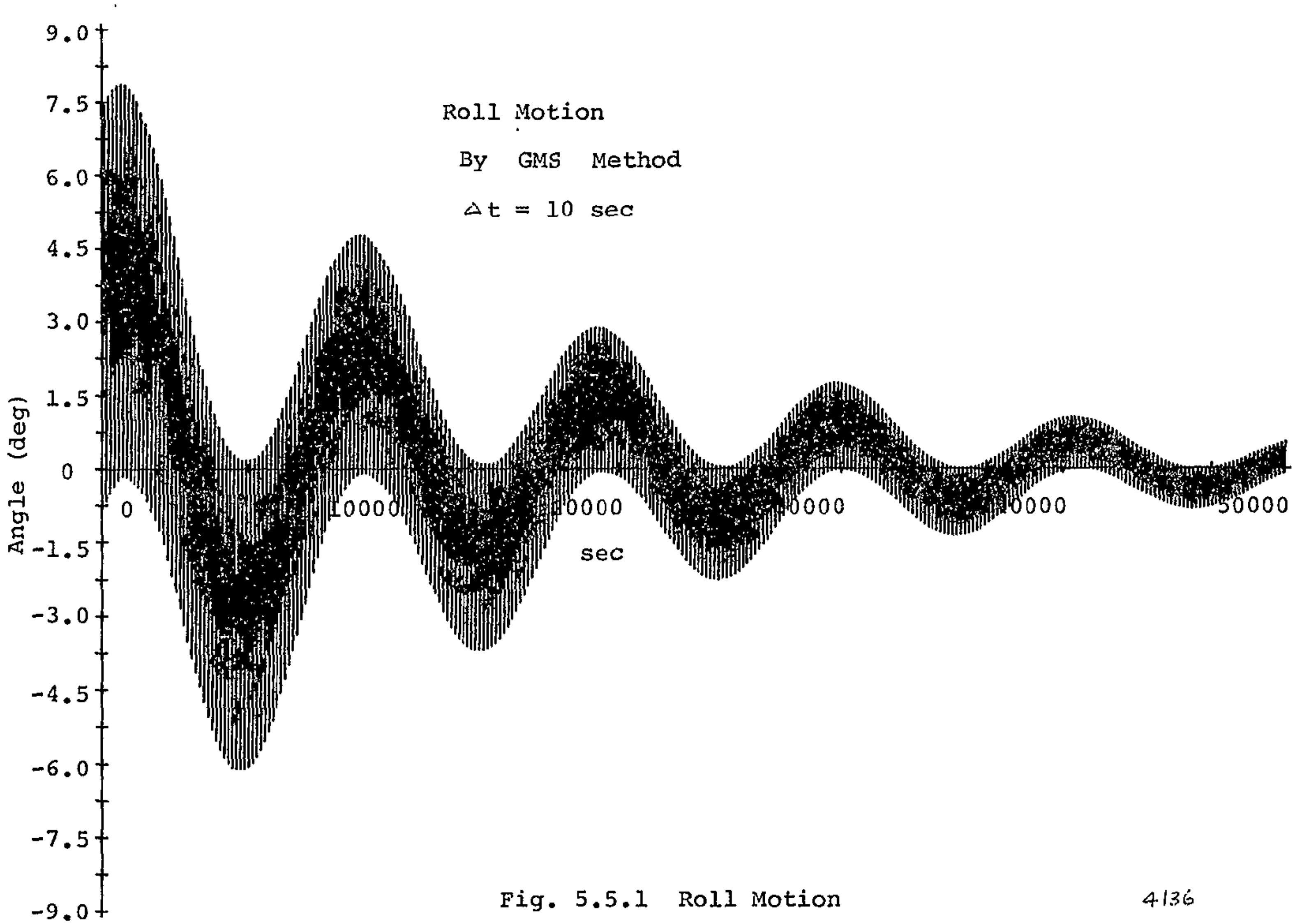


Fig. 5.5.1 Roll Motion

4136

C-3

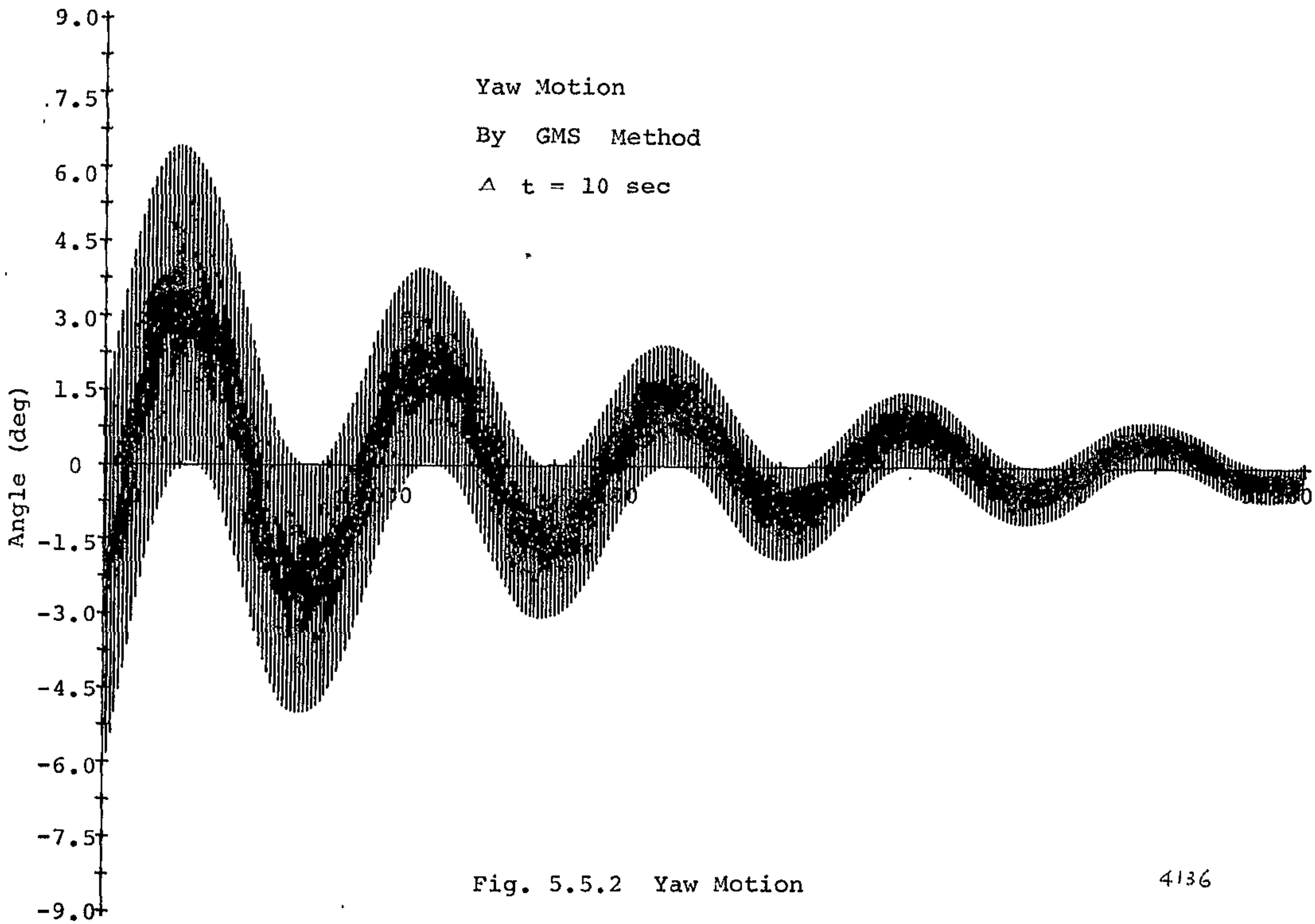


Fig. 5.5.2 Yaw Motion

4136

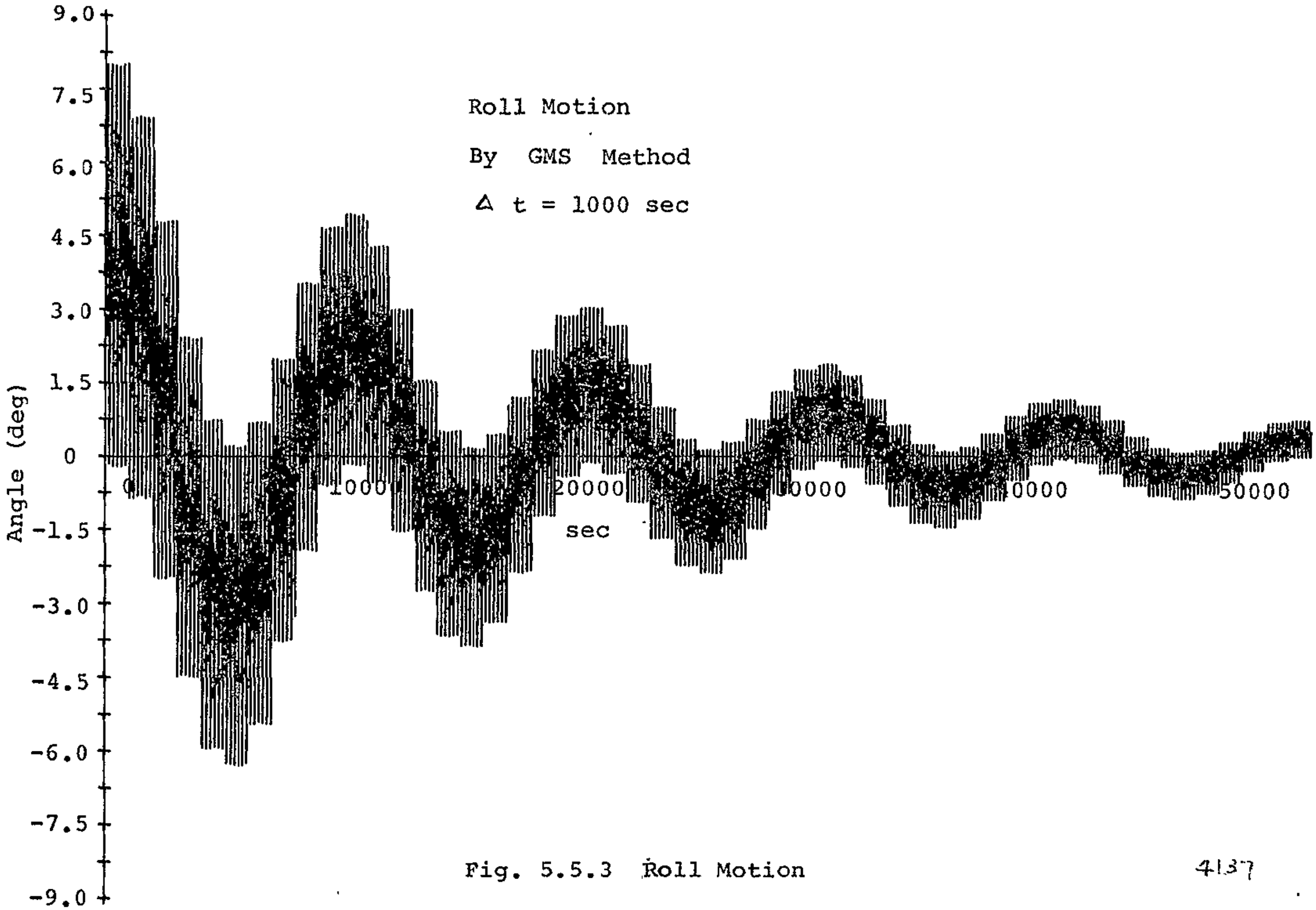


Fig. 5.5.3 Roll Motion

4137

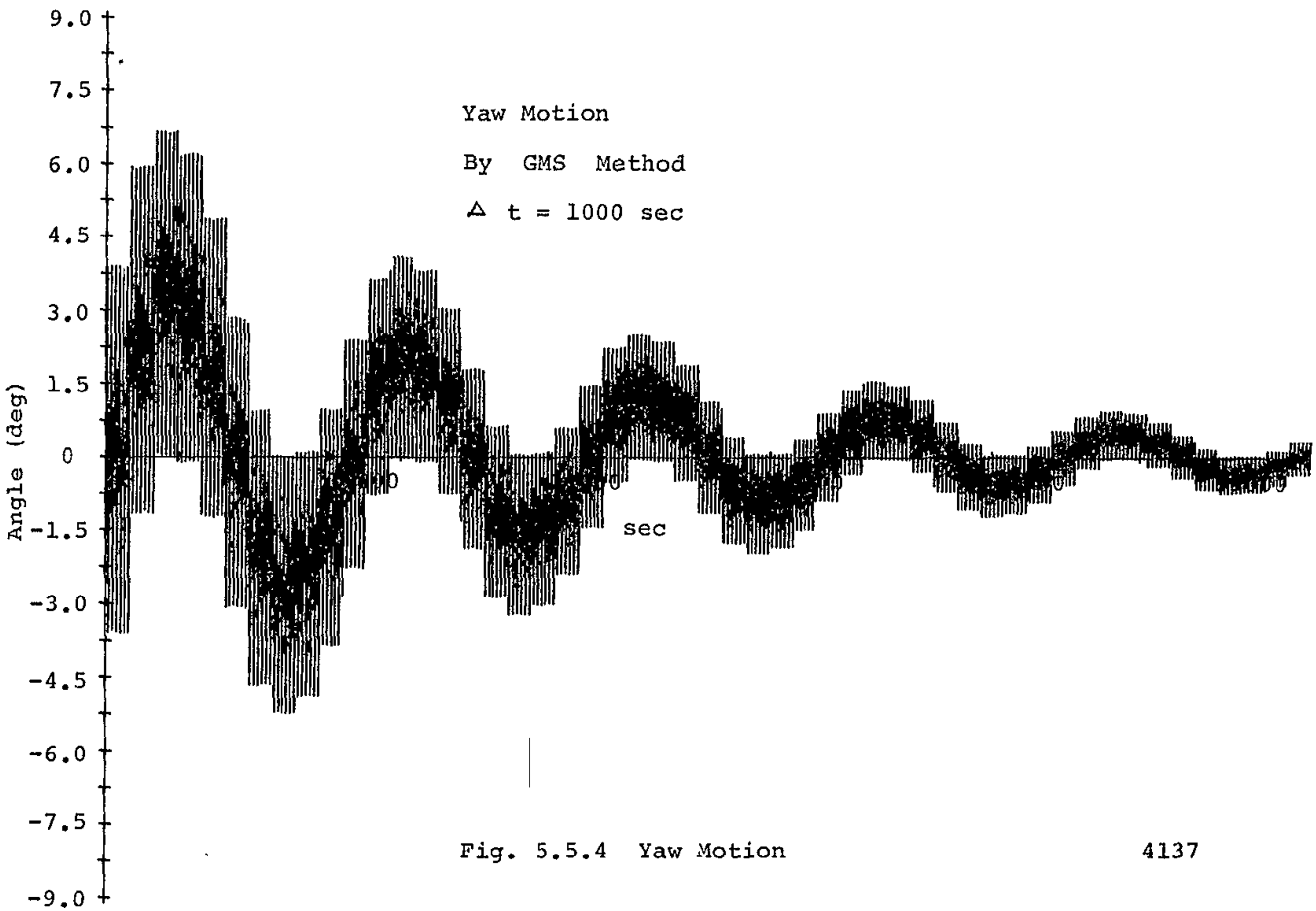


Fig. 5.5.4 Yaw Motion 4137

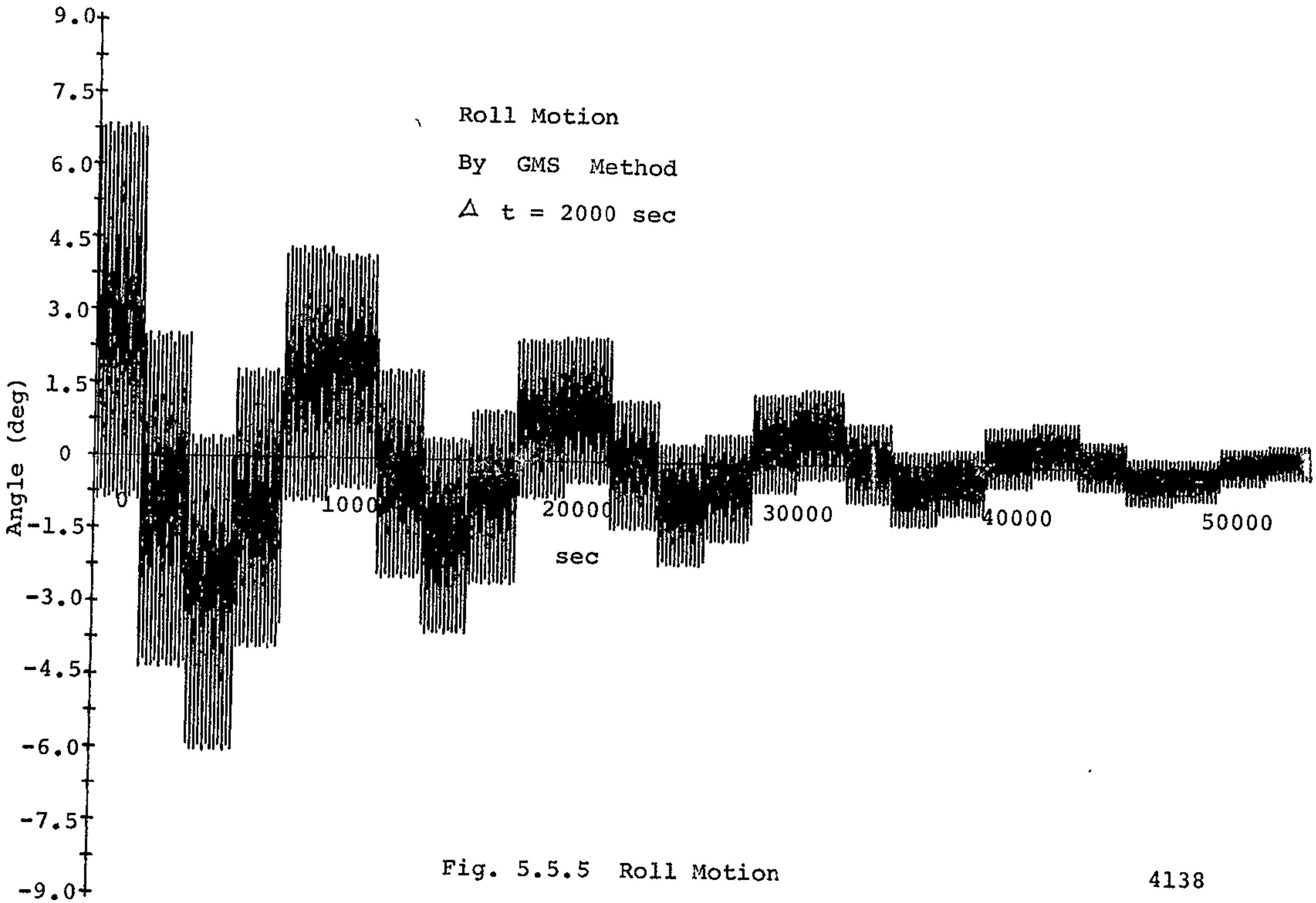


Fig. 5.5.5 Roll Motion

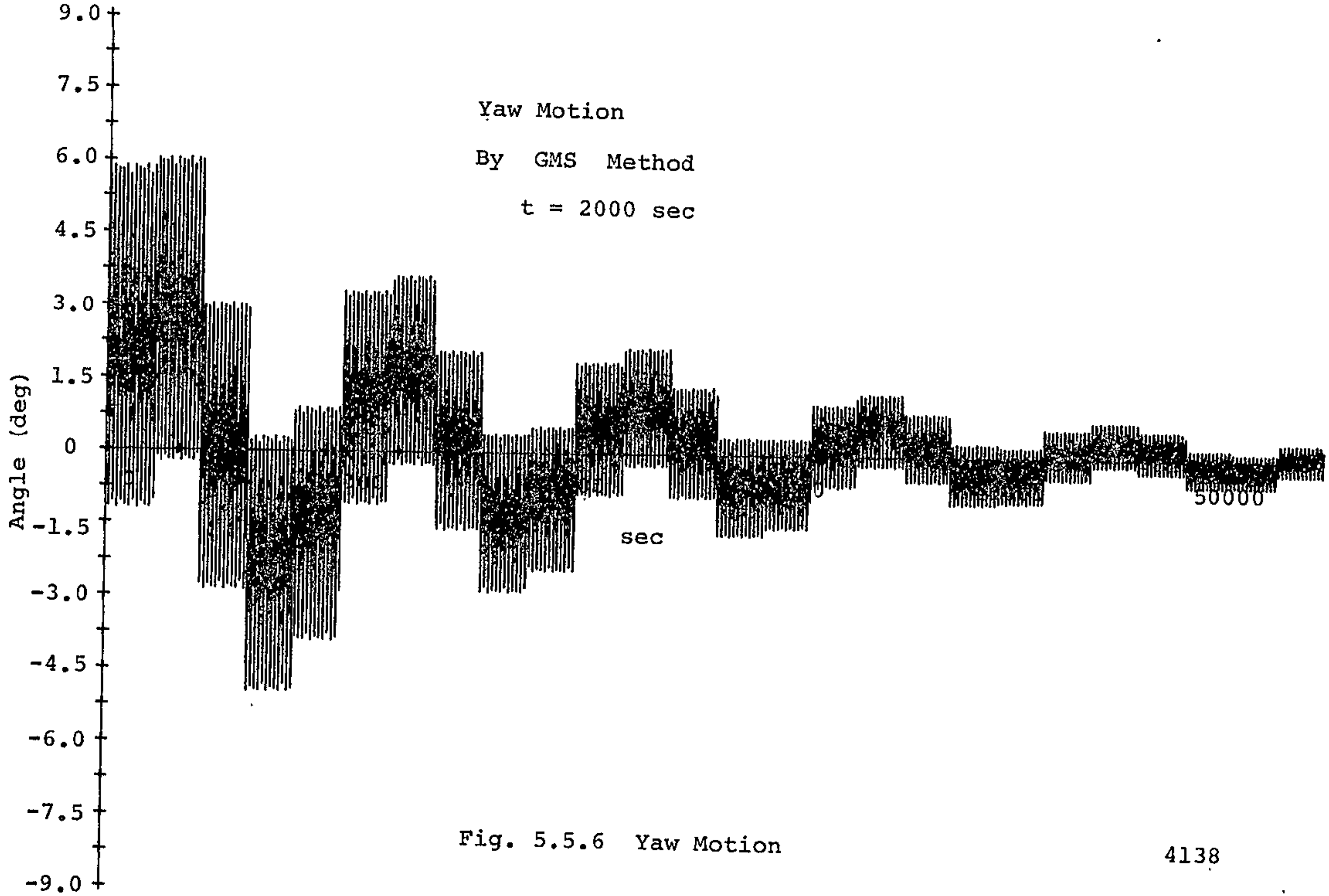
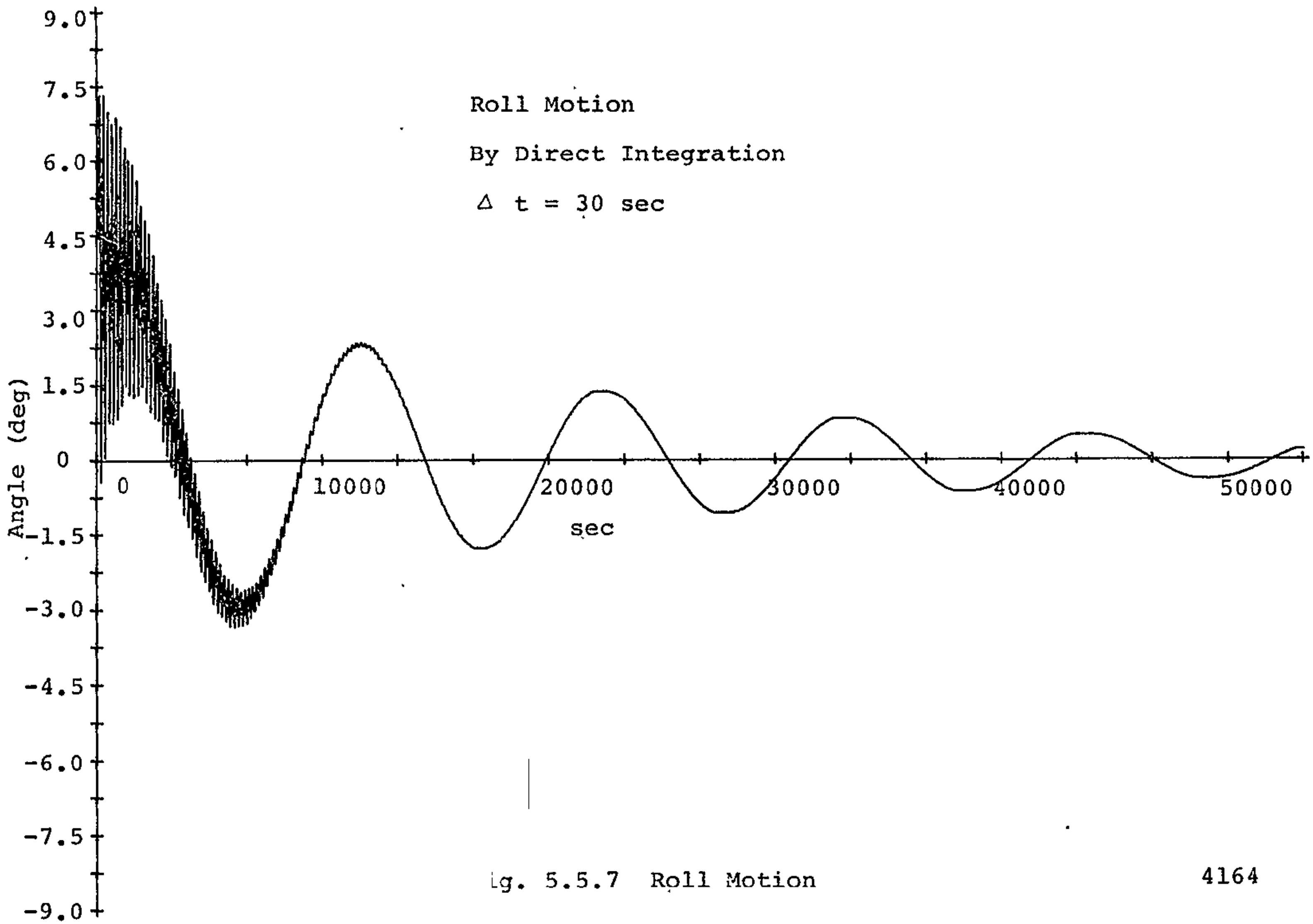


Fig. 5.5.6 Yaw Motion



ig. 5.5.7 Roll Motion

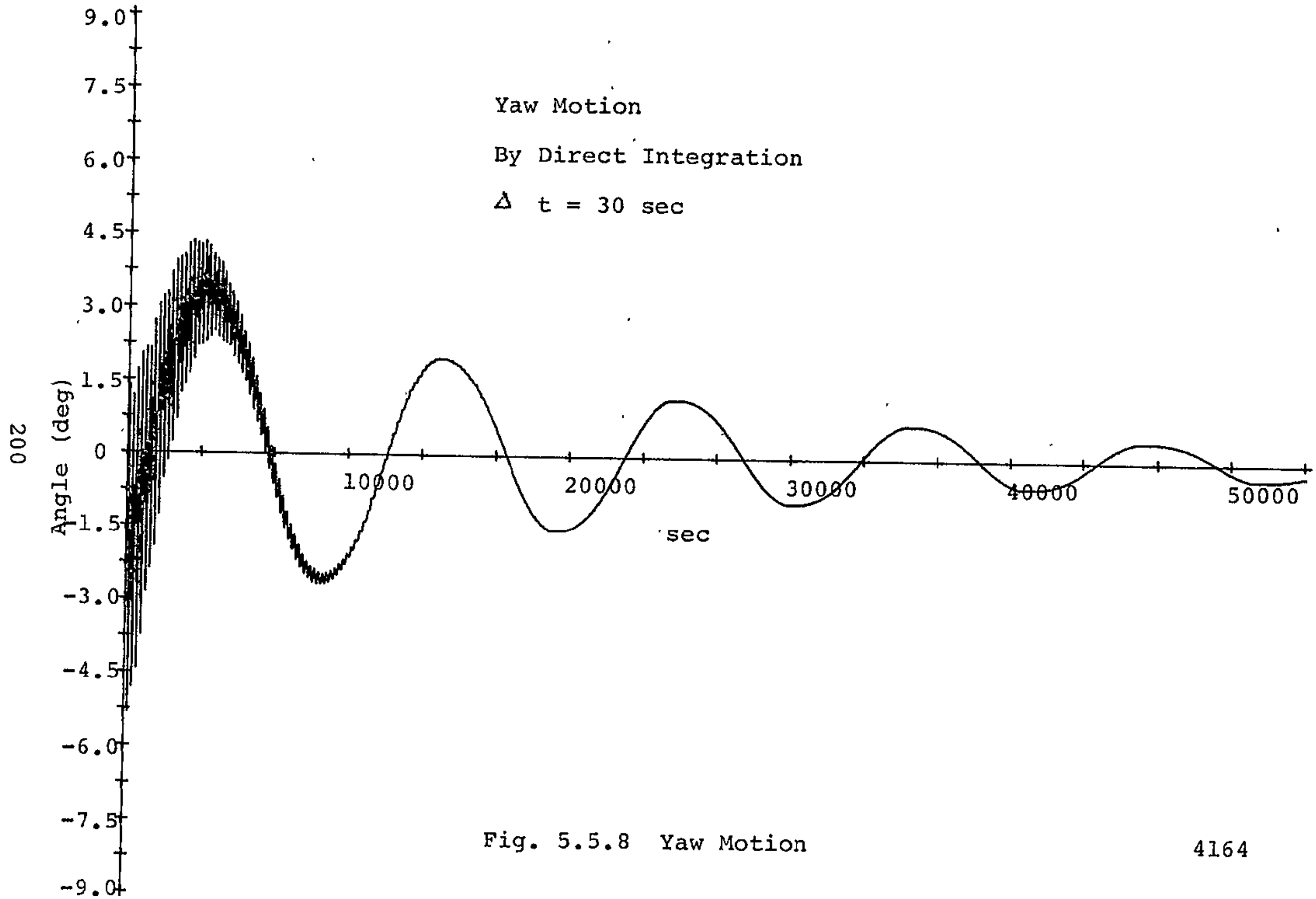


Fig. 5.5.8 Yaw Motion

4164

Table 3.6.1 Rigid Body Satellite Disturbed By Gravity Gradient Torque

Runs	Description	Computer Time* (sec)	Max. Errors		Figures
			$\bar{\omega}$ (rad/sec)	$\bar{\beta}$	
1	Direct Integration $\Delta T = 10$ sec (reference case)	106	0	0	
2	Asymptotic Approach $\Delta T = 10$ sec	288**	0.5×10^{-5}	.01	Fig. 3.6.1 3.6.2
3	Direct Integration $\Delta T = 50$ sec	22	1.2×10^{-5}	.08	Fig. 3.6.3 3.6.4
4	Asymptotic Approach $\Delta T = 100$ sec	31	1.0×10^{-5}	.01	Fig. 3.6.5 3.6.6
5	Asymptotic Approach $\Delta T = 200$ sec	16	1.0×10^{-5}	.01	Fig. 3.6.7 3.6.8
6	Asymptotic Approach $\Delta T = 500$ sec	6.5	2.0×10^{-5}	.03	Fig. 3.6.9 3.6.10
7	Torque-free Sol. Subtract Ref. Case (Effect of G.G.T.)		5.0×10^{-5}	.05	Fig. 3.6.11 3.6.12

* IBM 360/75 run time.

** Does not include initialization time - for nominal solution and Fourier transformations - in this case is about 40 sec.

Table 3.7.1 Rigid Body Satellite Disturbed By Geomagnetic Torque

Runs	Description	Computer Time (sec)	Max. Errors		Figures
			$\bar{\omega}$ (rad/sec)	$\bar{\beta}$	
1	Direct Integration $\Delta T = 10$ sec (reference case)	112	0	0	
2	Asymptotic Approach $\Delta T = 10$ sec	294	0.2×10^{-7}	.002	Fig. 3.7.1 3.7.2
3	Direct Integration $\Delta T = 25$ sec	45	3.0×10^{-7}	.005	Fig. 3.7.3 3.7.4
4	Asymptotic Approach $\Delta T = 100$ sec	31	0.2×10^{-7}	.002	Fig. 3.7.5 3.7.6
5	Asymptotic Approach $\Delta T = 200$ sec	16	$.25 \times 10^{-7}$	.003	Fig. 3.7.7 3.7.8
6	Asymptotic Approach $\Delta T = 500$ sec	6.5	$.30 \times 10^{-7}$	.003	Fig. 3.7.9 3.7.10
7	Torque-free Sol. Subtract Ref. Case (Effect of G.M.T.)		100×10^{-7}	.003	Fig. 3.7.11 3.7.12

Table 4.4.1 Dual Spin Satellite Disturbed by Gravity Gradient Torque

Runs	Description	Computer Time (sec)	Max. Errors $\bar{\omega}$ (rad/sec)	$\bar{\beta}$	Figures
1	Direct Integration $\Delta T = 10$ sec (reference case)	109	0	0	
2	Asymptotic Approach $\Delta T = 10$ sec	297	0.2×10^{-4}	.005	Fig. 4.4.1 4.4.2
3	Direct Integration $\Delta T = 50$ sec	22	10×10^{-4}	.20	FIG. 4.4.3 4.4.4
4	Asymptotic Approach $\Delta T = 100$ sec	31	0.3×10^{-4}	.006	FIG. 4.4.5 4.4.6
5	Asymptotic Approach $\Delta T = 500$ sec	6.5	0.5×10^{-4}	.008	FIG. 4.4.7 4.4.8
6	Torque-free Sol. Subtract Ref. Case (Effect of G.G.T.)		1.5×10^{-4}	.020	FIG. 4.4.9 4.4.10

Table 4.4.2 Dual Spin Satellite Disturbed By Geomagnetic Torque

Runs	Description	Computer Time (sec)	Max. Errors $\bar{\omega}$ (rad/sec)	Max. Errors $\bar{\beta}$	Figures
1	Direct Integration $\Delta T = 10$ sec (reference case)	102	0	0	
2	Asymptotic Approach $\Delta T = 10$ sec	290	0.5×10^{-5}	.0010	Fig. 4.4.11 4.4.12
3	Direct Integration $\Delta T = 25$ sec	42	10×10^{-5}	.0150	Fig. 4.4.13 4.4.14
4	Asymptotic Approach $\Delta T = 100$ sec	30	0.5×10^{-5}	.0015	Fig. 4.4.15 4.4.16
5	Asymptotic Approach $\Delta T = 500$ sec	6.4	0.6×10^{-5}	.0017	Fig. 4.4.17 4.4.18

Table 5.5 GMS Approximation*and Direct Solution

Case	Description	Computer time (sec)	Fig.
1	GMS Approximation Δt=1000 sec	2.9	5.5.3 5.5.4
2	GMS Approximation Δt=2000 sec	1.6	5.5.5 5.5.6
3	Direct integration Δt=10 sec	57.9	5.4.3 5.4.4
4	Direct integration Δt=30 sec	19.2	5.5.7 5.5.8

* General multiple scales with nonlinear clock.